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CERTAIN RESULTS OF RIEMANN-LIOUVILLE FRACTIONAL CALCULUS INVOLVING SEVEN-PARAMETRIC MITTAG-LEFFLER FUNCTION

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Abstract. This analysis aims to establish Riemann-Liouville derivation and integral operators regarding the recently suggested seven-parameter Mittag-Leffler function then investigates the corresponding special cases. In addition, certain notable results associated with those new operators have been discussed.

Keywords. Fractional calculus, Riemann-Liouville operators, Mittag-Leffler function.
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1. Introduction and preliminaries

Fractional calculus is a illustrious mathematical field that interested in fractional order of differentials and integrals. The origin of fractional calculus was invented by virtue of a question in the letters between Leibniz and L'Hopital in 1695, which is " Can the differential $d^n x$ has meaning when n is fraction?", then it was developed to the integration concept. Thereafter, many scientists were curios to discuss that new idea Such as Bernoulli, Wallis, Lagrange, Laplace, Liouville, and Riemann. In modern years, this domain has attracted considerable attention to achieve a vital properties and applications which employed in various sciences such as physics, biology, economic and finance as well as engineering, See[[1],[6], [8], [12], [21]].

Later, numerous different methods related to fractional calculus was raised, not being all equivalent, for instance the Riemann-Liouville fractional calculus which demonstrate common features to evaluating fractional differentials and integrals. For a well-behaved function $h(s)$ that defined over (σ, δ) with $s \in \mathbb{R}^+$, then for any order $\lambda > 0$, the left-sided and right-sided Riemann-Liouville fractional integral are specified as [7]

$$\left(I^{\lambda}_{\kappa^+} h\right)(s) = \frac{1}{\Gamma(\lambda)} \int_{\kappa}^{\omega} \frac{h(\omega)}{(s-\omega)^{1-\lambda}}\, d\omega, \tag{1.1}$$

$$\left(I^{\lambda}_{\kappa^-} h\right)(s) = \frac{1}{\Gamma(\lambda)} \int_{\omega}^{\kappa} \frac{h(\omega)}{(\omega - s)^{1-\lambda}}\, d\omega. \tag{1.2}$$

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Note that, the identity operator occurs when $\lambda = 0$, and the above integral possesses the semi-group property $\left(I_{\kappa^+}^{\lambda_1} I_{\kappa^+}^{\lambda_2} h\right)(s) = \left(I_{\kappa^+}^{\lambda_1+\lambda_2} h\right)(s)$ for $\lambda_1, \lambda_2 \geq 0$, as well as the composition property $\left(I_{\kappa^+}^{\lambda_1} I_{\kappa^+}^{\lambda_2} h\right)(s) = \left(I_{\kappa^+}^{\lambda_2} I_{\kappa^+}^{\lambda_1} h\right)(s)$.

On the other side, the Riemann-Liouville fractional differentials considered as the inverse operators of the associated integrals (1.1) and (1.2). Accordingly, the left-sides and right-sides Riemann-Liouville fractional differentials is considering as [7]

$$\left(D_{\kappa^+}^{\lambda} h\right)(s) = \frac{d^n}{ds^n} \left(I_{\kappa^+}^{\lambda} h\right)(s), \tag{1.3}$$

$$\left(D_{\kappa^-}^{\lambda} h\right)(s) = \frac{d^n}{ds^n} \left(I_{\kappa^-}^{\lambda} h\right)(s), \tag{1.4}$$

Notably

$$D^{\epsilon} y^{\xi} = \frac{\Gamma(\xi+1)}{\Gamma(\xi-\epsilon+1} \; y^{\xi-\epsilon}. \tag{1.5}$$

Among the most significant functions that were discussed in view of Riemann-Liouville fractional calculus was the Mittag-Leffler function

$$E_{\tau}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\tau n+1)}, \tag{1.6}$$

where z $\in \mathbb{C}$, and $Re(\tau) > 0$. This is function with no singularities in $\mathbb{C}$, with order $\frac{1}{Re(\tau)}$ and type 1, that generalizes the exponential function [15]. This function is considered the dominant field of fractional calculus due to its wide applications in that field, see [3, 9, 11, 13, 18, 19, 23]. By virtue of the importance of Mittag-Leffler function, multitudinous generalizations were suggested starting from the Wiman's function [22]

$$E_{\tau,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\tau n+\mu)}, \tag{1.7}$$

where, $z \in \mathbb{C}, Re(\tau) > 0$, and $Re(\mu) > 0$. Thereafter, wide area of generalization studies covered the Mittag-Leffler function with multitudinous characteristics and applications, see [1, 2, 4, 5, 10, 16, 17].

Recently, a seven-parameter Mittag-Leffler function was submitted by the authors [20], which is a globally holomorphic function with estimated order and type.

$$E_{\tau_1,\mu_1,\tau_2,\mu_2}^{\sigma,\delta,\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\sigma)_n \; (\delta)_n}{(\gamma)_n \;\; \Gamma(\tau_1 n+\mu_1) \; \Gamma(\tau_2 n+\mu_2) n!} \, z^n. \tag{1.8}$$

Moreover, the following important explicit formula had been realized:

$$E_{\tau_1,\mu_1,\tau_2,\mu_2}^{\sigma,\delta,\gamma}(z) = \frac{\Gamma(\gamma)}{\Gamma(\sigma)\Gamma(\delta)} \; {}_2\Psi_3 \left[\begin{array}{c} (\sigma,1),(\delta,1) \\ (\gamma,1),(\tau_1,\mu_1),(\tau_2,\mu_2) \end{array} \middle| \; z \right] \tag{1.9}$$

$$E_{\tau_1,\mu_1,\tau_2,\mu_2}^{\sigma,\delta,\gamma}(z) = \frac{\Gamma(\gamma)}{\Gamma(\sigma)\Gamma(\delta)} H_{2,4}^{1,2} \left[-z \middle| \begin{array}{c} (1-\sigma,1), \; (1-\delta,1) \\ (0,1), \; (1-\gamma,1), \; (1-\mu_1,\tau_1), \; (1-\mu_2,\tau_2) \end{array} \right] \tag{1.10}$$

where Ψ and H are the respective well-known Fox-wright function and H-Function [14].

As demonstrated above, the function (1.8) has numerous properties which could be beneficial for work directions of geometric function theory, canonical functions and many applications in mathematics, physics and engineering. Motivated by their significance, the present study investigates the Riemann-Liouville fractional integrals and differentials for the function (1.8) and survey major corresponding features virtually.

2. Main results

The core notion of our investigations, is to demonstrate certain Riemann-Liouville fractional integral and differential operators in consideration of seven-parameter Mittag-Leffler function (1.8), then consult the core features and relations involving these operators.

Initially, a recurrence relation was employed to establish more characterization related to the Rimann-Liouville fractional calculus operators for function (1.8).

Lemma 2.1.

$$r\omega^{\tau_1}\; E^{\sigma,\delta,\gamma}_{\tau_1,\mu_1,\tau_2,\mu_2}(r\omega^{\tau_1}) = \frac{\gamma-1}{\delta-1}\left[E^{\sigma,\delta-1,\gamma-1}_{\tau_1,\mu_1-\tau_1,\tau_2,\mu_2-\tau_2}\;(r\omega^{\tau_1}) - E^{\sigma-1,\delta-1,\gamma-1}_{\tau_1,\mu_1-\tau_1,\tau_2,\mu_2-\tau_2}\;(r\omega^{\tau_1})\right]. \tag{2.1}$$

Proof. Utilizing the Pochhammer characteristic that $(n+1)(\sigma)_n = (\sigma)_{n+1} - (\sigma-1)_{n+1}$ on the left-hand side of relation (2.1), we achieve:

$$= \sum_{n=0}^{\infty} \frac{(\sigma)_{n+1}\;(\delta)_n}{(\gamma)_n\;\;\Gamma(\tau_1 n+\mu_1)\Gamma(\tau_2 n+\mu_2)(n+1)!}\;(r\omega^{\tau_1})^{n+1}$$
$$-\sum_{n=0}^{\infty} \frac{(\sigma-1)_{n+1}\;(\delta)_n}{(\gamma)_n\;\Gamma(\tau_1 n+\mu_1)\;\Gamma(\tau_2 n+\mu_2)(n+1)!}\;(r\omega^{\tau_1})^{n+1}.$$

which gives,

$$= \frac{\gamma-1}{\delta-1}\sum_{n=0}^{\infty} \frac{(\sigma)_{n+1}\;(\delta-1)_{n+1}}{(\gamma-1)_n\;\;\Gamma(\tau_1(n+1)+\mu_1-\tau_1)\Gamma(\tau_2(n+1)+\mu_2-\tau_2)(n+1)!}\;(r\omega^{\tau_1})^{n+1}$$
$$-\frac{\gamma-1}{\delta-1}\sum_{n=0}^{\infty} \frac{(\sigma-1)_{n+1}\;(\delta-1)_{n+1}}{(\gamma-1)_{n+}\;\Gamma(\tau_1(n+1)+\mu_1-\tau_1)\;\Gamma(\tau_2(n+1)+\mu_2-\tau_2)(n+1)!}\;(r\omega^{\tau_1})^{n+1}.$$

as was to be required. □

Now, we observe left-sided Riemann-Liouville fractional integral operator in terms of the function (1.8).

Theorem 2.2. *Let* $\lambda > 0, r \in \mathbb{R},\;\; \tau_1,\tau_2 \in \mathbb{C}$ *and* $\min\{Re(\sigma), Re(\delta), Re(\gamma), Re(\mu_1), Re(\mu_2)\} > 0$. *Then*

$$\left(I^{\lambda}_{0+}\left[\omega^{\mu_1-1}E^{\sigma,\delta,\gamma}_{\tau_1,\mu_1,\tau_2,\mu_2}(r\omega^{\tau}_1)\right]\right)(y) = y^{\mu_1+\lambda-1}E^{\sigma,\delta,\gamma}_{\tau_1,\mu_1+\lambda,\tau_2,\mu_2}(ry^{\tau_1}), \tag{2.2}$$

where I^{λ}_{0+} *is the Riemann-Liouville left-sided fractional integral operator* (1.1).

Proof. In view of expressions (1.8) and (1.1), we consider

$$\Psi = \left(I^{\lambda}_{0+}\left[\omega^{\mu_1-1}E^{\sigma,\delta,\gamma}_{\tau_1,\mu_1,\tau_2,\mu_2}(r\omega^{\tau}_1)\right]\right)(y)$$
$$= \frac{\omega^{\mu_1-1}}{\Gamma(\lambda)}\int_0^y\;(y-\omega)^{\lambda-1}\sum_{n=0}^{\infty}\frac{(\sigma)_n\;(\delta)_n}{(\gamma)_n\;\;\Gamma(\tau_1 n+\mu_1)\;\Gamma(\tau_2 n+\mu_2)n!}\;r^n\omega^{n\tau_1}d\omega,$$

substituting $\omega = sy$ and simple arranging involving the definition of beta function, we realize that

$$\Psi = \frac{r^n y^{n\tau_1+\mu_1+\lambda-1}}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{(\sigma)_n \ (\delta)_n}{(\gamma)_n \ \ \Gamma(\tau_1 n+\mu_1) \ \Gamma(\tau_2 n+\mu_2) n!} \ \delta(\lambda, n\tau_1+\mu_1),$$
$$= y^{\mu_1+\lambda-1} \ E^{\sigma,\delta,\gamma}_{\tau_1,\mu_1+\lambda,\tau_2,\mu_2}(r\omega^{\tau_1}).$$

□

Particular cases. One can directly observe from [**?**], that the following operators hold as respective particular cases of the function (1.8).

- The standard Mittag-Leffler function:

$$\left(I^{\lambda}_{0^+}\left[E^{1,\delta,\delta}_{\tau_1,1,0,1}(r\omega^{\tau_1})\right]\right)(y) = y^{\lambda} E_{\tau_1}(ry^{\tau_1}),$$

- Wiman's function:

$$\left(I^{\lambda}_{0^+}\left[\omega^{\mu_1-1} E^{1,\delta,\delta}_{\tau_1,\mu_1,0,1}(r\omega^{\tau_1})\right]\right)(y) = y^{\mu_1+\lambda-1} E_{\tau_1,\mu_1+\lambda}(ry^{\tau_1}),$$

- Prabhakar's three-parameter Mittag-Leffler function:

$$\left(I^{\lambda}_{0^+}\left[\omega^{\mu_1-1} E^{\sigma,\delta,\delta}_{\tau_1,\mu_1,0,1}(r\omega^{\tau_1})\right]\right)(y) = y^{\mu_1+\lambda-1} E^{\sigma}_{\tau_1,\mu_1+\lambda}(ry^{\tau_1}),$$

- Dherbashian's four-parameter Mittag-Leffler function:

$$\left(I^{\lambda}_{0^+}\left[\omega^{\mu_1-1} E^{1,\delta,\delta}_{\tau_1,\mu_1,\tau_2,\mu_2}(r\omega^{\tau_1})\right]\right)(y) = y^{\mu_1+\lambda-1} E_{\tau_1,\mu_1+\lambda,\tau_2,\mu_2}(ry^{\tau_1}),$$

- Salim's four-parameter Mittag-Leffler function:

$$\left(I^{\lambda}_{0^+}\left[\omega^{\mu_1-1} E^{\sigma,1,\gamma}_{\tau_1,\mu_1,1,1}(r\omega^{\tau_1})\right]\right)(y) = y^{\mu_1+\lambda-1} E^{\sigma,\gamma}_{\tau_1,\mu_1+\lambda}(ry^{\tau_1}),$$

- Five-parameter Mittag-Leffler function:

$$\left(I^{\lambda}_{0^+}\left[\omega^{\mu_1-1} E^{\sigma,\delta,\delta}_{\tau_1,\mu_1,\tau_2,\mu_2}(r\omega^{\tau_1})\right]\right)(y) = y^{\mu_1+\lambda-1} E^{\sigma}_{\tau_1,\mu_1+\lambda,\tau_2,\mu_2}(ry^{\tau_1}),$$

- Wright function:

$$\left(I^{\lambda}_{0^+}\left[E^{\sigma,1,\gamma}_{\tau_1,\mu_1,1,1}(r\omega^{\tau_1})\right]\right)(y) = y^{\mu_1+\lambda-1} \ W_{\tau_1,\mu_1+\lambda}(ry^{\tau_1}),$$

- Gaussian hypergeometric function:

$$\left(I^{\lambda}_{0^+}\left[E^{\sigma,\delta,\delta}_{0,1,0,1}(r\omega^{\tau_1})\right]\right)(y) = y^{\lambda} \ {}_2F_1(\sigma,\delta;\gamma;ry^{\tau_1}),$$

- Kummer function:

$$\left(I^{\lambda}_{0^+}\left[E^{\sigma,1,\gamma}_{1,1,0,1}(r\omega^{\tau_1})\right]\right)(y) = y^{\lambda} \ {}_1F_1(\sigma,\gamma;ry^{\tau_1}).$$

Corollary 2.3. *Let* $\lambda > 0, r \in \mathbb{R}, \ \tau_1, \tau_2 \in \mathbb{C}$ *and* $\min\{Re(\sigma), Re(\delta), Re(\gamma), Re(\mu_1), Re(\mu_2)\} > 0$. *Then*

$$\left(I^{\lambda}_{0^+}\left[\omega^{\mu_1-1} E^{\sigma,\delta,\gamma}_{\tau_1,\mu_1,\tau_2,\mu_2}(r\omega^{\tau_1})\right]\right)(y) = \frac{(\gamma-1) \ y^{\lambda+\mu_1-1}}{\delta-1}$$
$$\left[E^{\sigma,\delta-1,\gamma-1}_{\tau_1,\mu_1-\tau_1+\lambda,\tau_2,\mu_2-\tau_2} \ (ry^{\tau_1}) - E^{\sigma-1,\delta-1,\gamma-1}_{\tau_1,\mu_1-\tau_1+\lambda,\tau_2,\mu_2-\tau_2} \ (ry^{\tau_1})\right].$$

Corollary 2.4. *Let* $\lambda > 0, r \in \mathbb{R},\ \tau_1, \tau_2 \in \mathbb{C}$ *and* $\min\{Re(\sigma), Re(\delta), Re(\gamma), Re(\mu_1), Re(\mu_2)\} > 0$. *Then*

$$\left(I_{0+}^{\lambda}\left[\omega^{\mu_1-1}E_{\tau_1,\mu_1,\tau_2,\mu_2}^{\sigma,\delta,\gamma}(r\omega_1^{\tau})\right]\right)(y) = \frac{\Gamma(\gamma)}{\Gamma(\sigma)\Gamma(\delta)}y^{\mu_1+\lambda-1} \tag{2.3}$$

$$H_{2,4}^{1,2}\left[-z\left|\begin{matrix}(1-\sigma,1),\ (1-\delta,1)\\(0,1),\ (1-\gamma,1),\ (1-\mu_1+\lambda,\tau_1),\ (1-\mu_2,\tau_2)\end{matrix}\right.\right]. \tag{2.4}$$

Next, we proceed to achieve the right-sided fractional integral Riemann-Liouville operator for the function (1.8) with some consequences outcomes.

Proposition 2.5. *Let* $\lambda > 0, r \in \mathbb{R},\ \tau_1, \tau_2 \in \mathbb{C}$ *and* $\min\{Re(\sigma), Re(\delta), Re(\gamma), Re(\mu_1), Re(\mu_2)\} > 0$. *Then*

$$\left(I_{-}^{\lambda}\left[\omega^{\mu_1-1}E_{\tau_1,\mu_1,\tau_2,\mu_2}^{\sigma,\delta,\gamma}(r\omega_1^{\tau})\right]\right)(y) = y^{\lambda+\mu_1-1}E_{\tau_1,\mu_1+\lambda,\tau_2,\mu_2}^{\sigma,\delta,\gamma}(ry^{\tau_1}), \tag{2.5}$$

where I_{-}^{λ} *is the Riemann-Liouville right-sided fractional integral operator* (1.2).

Proof. Similarly as the proof of Theorem 2.2, in view of expressions (1.8) and (1.2), and basic computations involving the definition of Generalized beta function, we get the acquired result. □

Remark 2.6. For necessity, we mention that the relations (2.1) and (1.9) are holds for the operator (2.5) similarly as the previous corollaries.

The following outcomes discuss another features for calculus of non-integer order associated with (1.8) involving the operator of Riemann-Liouville fractional differentials D_{0+}^{λ} and D_{-}^{λ} respectively.

Theorem 2.7. *Let* $\lambda > 0, r \in \mathbb{R},\ \tau_1, \tau_2 \in \mathbb{C}$ *and* $\min\{Re(\sigma), Re(\delta), Re(\gamma), Re(\mu_1), Re(\mu_2)\} > 0$. *Then*

$$\left(D_{0+}^{\lambda}\left[\omega^{\mu_1-1}E_{\tau_1,\mu_1,\tau_2,\mu_2}^{\sigma,\delta,\gamma}(r\omega_1^{\tau})\right]\right)(y) = y^{\mu_1-\lambda-1}E_{\tau_1,\mu_1-\lambda,\tau_2,\mu_2}^{\sigma,\delta,\gamma}(ry^{\tau_1}), \tag{2.6}$$

where D_{0+}^{λ} *is the Riemann-Liouville left-sided fractional differential operator* (1.3).

Proof. In view of expressions (1.8) and (1.1), we consider

$$\begin{aligned}\Phi &= \left(D_{0+}^{\lambda}\left[\omega^{\mu_1-1}E_{\tau_1,\mu_1,\tau_2,\mu_2}^{\sigma,\delta,\gamma}(r\omega_1^{\tau})\right]\right)(y)\\ &= \left(\frac{d}{dy}\right)^{[\lambda]+1}\left(I_{0+}^{1-\{\lambda\}}\left[\omega^{\mu_1-1}E_{\tau_1,\mu_1,\tau_2,\mu_2}^{\sigma,\delta,\gamma}(r\omega_1^{\tau})\right]\right)(y)\\ &= \left(\frac{d}{dy}\right)^{[\lambda]+1}\frac{\omega^{\mu_1-1}}{\Gamma(1-\{\lambda\})}\\ &\qquad\int_0^y (y-\omega)^{\{\lambda\}}\sum_{n=0}^{\infty}\frac{(\sigma)_n\ (\delta)_n}{(\gamma)_n\ \ \Gamma(\tau_1 n+\mu_1)\ \Gamma(\tau_2 n+\mu_2)n!}\ r^n\omega^{n\tau_1}d\omega,\end{aligned}$$

substituting $\omega = sy$ and simple arranging involving the definition of beta function, we realize that

$$\begin{aligned}\Phi &= \sum_{n=0}^{\infty} \frac{(\sigma)_n\ (\delta)_n\ r^n}{(\gamma)_n\ \ \Gamma(\tau_1 n+\mu_1)\ \Gamma(\tau_2 n+\mu_2)\ \Gamma\left(1-\{\lambda\}\right)\ n!}\ y^{n\tau_1+\mu_1-\lambda-1} \\ &\qquad\qquad\qquad\qquad \delta\left(1-\{\lambda\}, n\tau_1+\mu_1\right), \\ &= y^{\mu_1+\lambda-1}\ E^{\sigma,\delta,\gamma}_{\tau_1,\mu_1-\lambda,\tau_2,\mu_2}(r\omega^{\tau_1}).\end{aligned}$$

□

Corollary 2.8. *Let* $\lambda > 0, r \in \mathbb{R},\ \tau_1, \tau_2 \in \mathbb{C}$ *and* $\min\{Re(\sigma), Re(\delta), Re(\gamma), Re(\mu_1), Re(\mu_2)\} > 0$. *Then*

$$\left(D^{\lambda}_{0+}\left[\omega^{\mu_1-1}E^{\sigma,\delta,\gamma}_{\tau_1,\mu_1,\tau_2,\mu_2}(r\omega^{\tau}_1)\right]\right)(y) = \left(\frac{d}{dy}\right)^{[\lambda]+1}\left(I^{1-\{\lambda\}}_{0+}\left[\omega^{\mu_1-1}E^{\sigma,\delta,\gamma}_{\tau_1,\mu_1,\tau_2,\mu_2}(r\omega^{\tau}_1)\right]\right)(y) \tag{2.7}$$

$$= \frac{(\gamma-1)\ y^{\lambda+\mu_1-1}}{\delta-1}\left[E^{\sigma,\delta-1,\gamma-1}_{\tau_1,\mu_1-\tau_1-\lambda,\tau_2,\mu_2-\tau_2}\ (ry^{\tau_1}) - E^{\sigma-1,\delta-1,\gamma-1}_{\tau_1,\mu_1-\tau_1-\lambda,\tau_2,\mu_2-\tau_2}\ (ry^{\tau_1})\right]. \tag{2.8}$$

Corollary 2.9. *Let* $\lambda > 0, r \in \mathbb{R},\ \tau_1, \tau_2 \in \mathbb{C}$ *and* $\min\{Re(\sigma), Re(\delta), Re(\gamma), Re(\mu_1), Re(\mu_2)\} > 0$. *Then*

$$\left(D^{\lambda}_{0+}\left[\omega^{\mu_1-1}E^{\sigma,\delta,\gamma}_{\tau_1,\mu_1,\tau_2,\mu_2}(r\omega^{\tau}_1)\right]\right)(y) = \left(\frac{d}{dy}\right)^{[\lambda]+1}\left(I^{1-\{\lambda\}}_{0+}\left[\omega^{\mu_1-1}E^{\sigma,\delta,\gamma}_{\tau_1,\mu_1,\tau_2,\mu_2}(r\omega^{\tau}_1)\right]\right)(y) \tag{2.9}$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\sigma)\Gamma(\delta)}y^{\mu_1+\lambda-1}\ H^{1,2}_{2,4}\left[-z\left|\begin{matrix}(1-\sigma,1),\ (1-\delta,1)\\ (0,1),\ (1-\gamma,1),\ (1-\mu_1-\lambda,\tau_1),\ (1-\mu_2,\tau_2)\end{matrix}\right.\right]. \tag{2.10}$$

Next theorem scrutinize right-sided Riemann-Liouville fractional differential operator pertaining to the function (1.8).

Theorem 2.10. *Let* $\lambda > 0, r \in \mathbb{R},\ \tau_1, \tau_2 \in \mathbb{C}$ *and* $\min\{Re(\sigma), Re(\delta), Re(\gamma), Re(\mu_1), Re(\mu_2)\}\ > 0$. *Then*

$$\left(D^{\lambda}_{-}\left[\omega^{\lambda-\mu_1}E^{\sigma,\delta,\gamma}_{\tau_1,\mu_1,\tau_2,\mu_2}(r\omega^{\tau_1})\right]\right)(y) = y^{-\mu_1}E^{\sigma,\delta,\gamma}_{\tau_1,\mu_1-\lambda,\tau_2,\mu_2}(r\omega^{\tau_1}) \tag{2.11}$$

where D^{λ}_{-} *is the Riemann-Liouville right-sided fractional differential operator* (1.4).

Proof. Consider

$$\begin{aligned}\Omega &= \left(D^{\lambda}_{-}\left[\omega^{\lambda-\mu_1}E^{\sigma,\delta,\gamma}_{\tau_1,\mu_1,\tau_2,\mu_2}(r\omega^{-\tau_1})\right]\right)(y) \\ &= \left(-\frac{d}{dy}\right)^{[\lambda]+1}\left(I^{1-\{\lambda\}}_{-}\left[\omega^{\lambda-\mu_1}E^{\sigma,\delta,\gamma}_{\tau_1,\mu_1,\tau_2,\mu_2}(r\omega^{-\tau_1})\right]\right)(y) \\ &= \left(-\frac{d}{dy}\right)^{[\lambda]+1}\frac{\omega^{\lambda-\mu_1}}{\Gamma\left(1-\{\lambda\}\right)} \\ &\qquad \int_y^{\infty}(\omega-y)^{-\{\lambda\}}\sum_{n=0}^{\infty}\frac{(\sigma)_n\ (\delta)_n}{(\gamma)_n\ \ \Gamma(\tau_1 n+\mu_1)\ \Gamma(\tau_2 n+\mu_2)n!}r^n\omega^{-n\tau_1}d\omega,\end{aligned}$$

Substitute $\omega = \frac{y}{s}$, which gives

$$\Omega = \sum_{n=0}^{\infty} \frac{(\sigma)_n \ (\delta)_n \ \ r^n}{(\gamma)_n \ \ \Gamma(\tau_1 n + \mu_1) \ \Gamma(\tau_2 n + \mu_2) \ \Gamma\left(1 - \{\lambda\}\right) n!} \left(-\frac{d}{dy}\right)^{[\lambda]+1} y^{\lambda - \{\lambda\} - \mu_1 - n\tau_1 + 1}$$

$$\int_0^1 s^{n\tau_1 + \mu_1 - \lambda + \{\lambda\} - 2} \ (1-s)^{-\{\lambda\}} dv.$$

Utilizing the beta function and the rule (1.5), and simple arithmetic, we gain the desired outcome. □

Remark 2.11. It should be declared that the relations (2.1) and (1.9) are holds for the operator (2.11) similarly as the previous corollaries.

Remark 2.12. It is worth mentioning that the aforementioned operators in Theorem 3, Theorem 4 and Theorem 5 have the same special cases as well as the operator in Theorem 1.

Remark 2.13. It is crucial to point out that all the previous results holds when we adopt the parameters μ_2 and τ_2 by means of the determinations.

3. Conclusion

This study was concerned to outline fractional calculus operators of Riemann-Liouville type considering Mittag-Leffler function with seven complex-valued parameters with certain special cases. Moreover, we gain a certain relations between those operators and certain well-known special functions involving the seven-parametric Mittag-Leffler function. All the presented operators are significant for many branches of sciences and may have suitable applications in kinetic equations as a future work.

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