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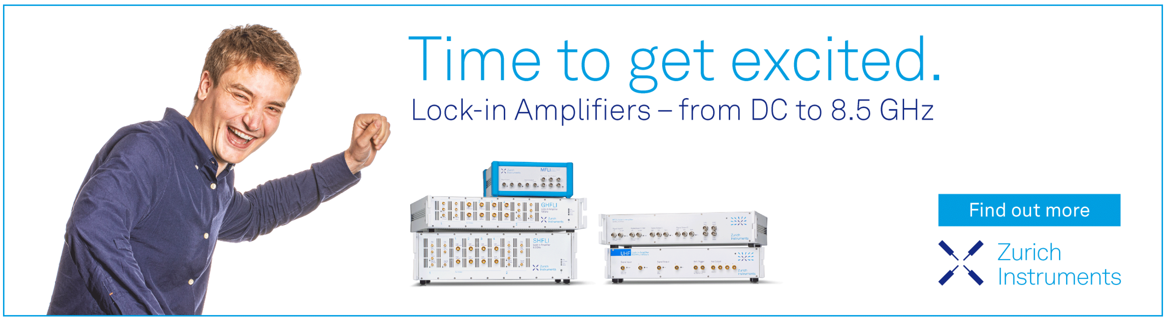

A New Class of K-Uniformly Starlike Functions Imposed by Generalized Salagean's Operator

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Abstract. Recently, numerous the generalizations of Hurwitz-Lerch zeta functions are investigated and introduced. In this paper, by using the extended generalized Hurwitz-Lerch zeta function, a new Salagean's differential operator is studied. Based on this new operator, a new geometric class and yielded coefficient bounds, growth and distortion result, radii of convexity, star-likeness, close-to-convexity, as well as extreme points are discussed.

Keywords. Univalent function; star-like function, convolution product; Salagean's differential operator; Hurwitz–Lerch zeta function.

INTRODUCTION

Let H be the class of analytic functions in the open unit disk $\Delta = \{z \in C : |z| < 1\}$ of the complex domain C.

Let $\mathrm{H}[a,m] = \{g \in \mathrm{H} : g(z) = a + a_m z^m + a_{m+1} z^{m+1} + ...\}$ and suppose that $\mathrm{H}_0 \equiv \mathrm{H}[0,1]$ and $\mathrm{H}_1 \equiv \mathrm{H}[1,1]$.

Let Γ denote the class of all normalized analytic functions g in Δ of the following formula,

$$g(z) = z + \sum_{m=2}^{\infty} a_m z^m, \qquad (z \in \Delta). \tag{1}$$

For two functions $g_l \in \Gamma$ are given by:

$$g_l(z) = z + \sum_{m=2}^{\infty} a_{m,l} z^m, l = 1,2, \quad (z \in \Delta), \tag{2}$$

the convolution is symbolized by $g_1 * g_2$ and shown as

$$(g_1 * g_2)(z) = z + \sum_{m=2}^{\infty} a_{m,1} a_{m,2} z^m.$$

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A function g in G is said to be star-like of order β in Δ if

$$Re\left\{\frac{zg'(z)}{g(z)}\right\}>\beta,$$

for some $\beta(0\le\beta<1)$ and for all $z\in\Delta.$ If $g(z)\in G$ satisfies the inequality

$$Re\left\{1+\frac{zg''(z)}{g'(z)}\right\}>\beta,$$

for some $\beta(0\le\beta<1)$ and for all $z\in\Delta,$ the $g(z)$ is to be convex of order β in Δ. On the other hand if $g(z)\in G$ satisfies the inequality

$$Re\{g'(z)\}>\beta,$$

for some $\beta(0\le\beta<1)$ and for all $z\in\Delta$, the $g(z)$ is said to be closed-to-convex of order β in Δ. It follows at once that $g(z)\in G$ is convex of order β if and only if $zg'(z)$ is star-like of order β in $\Delta,$ [1].

Let $S^*(\alpha)$ denote the class of star-like functions of order α such that $0\le\alpha<1$. A function $g\in S^*(\alpha)$ if

$$Re\left\{\frac{z\,g'(z)}{g(z)}\right\}\ge\alpha,(z\in\Delta).$$

It is well known that any functions in $S^*(\alpha)$ are univalent in $\Delta.$

The class S^* of star-like functions is identical by $S^*\equiv S^*(0).$ Bharti et al. [2] defined $\kappa-S_p(\alpha)$ to be the class of functions g with $0\le\alpha<1$ and $0\le\kappa<\infty$ the satisfy the condition:

$$Re\left\{\frac{z\,g'(z)}{g(z)}\right\}\ge\kappa\left|\frac{z\,g'(z)}{g(z)}-1\right|+\alpha. \tag{3}$$

This class $\kappa-S_p(\alpha)$ was studied in [3]. Most researchers are motivated to study and they introduced various classes of uniformly star-like and convex functions. For instance, Breaz [4], Breaz el. at [5], Stanciu and Breaz in [6], Al-Janaby et. al [7].

On the other hand, during the last century, the use of special functions (SFs) has been intensified fruitfully due to its importance in the Geometric Function Theory (GFT). The reason for attracting the authors towards SFs is that the class of hypergeometric functions was employed as a tool for resolving Bieberbach's problem in 1984 by De Branges, [8]. Afterward, numerous significant works on connections between analytic univalent and SFs have been discussed by several complex analysis such as, Mahmoud et. al [9], Atshan et. al [10], Elhaddad and Darus [11], Yan and Liu [12], Al-Janaby et. al [13], Oros [14] and Layth et. al [15].

The familiar Hurwitz–Lerch zeta function $\Phi(z,s,v)$ is shown by [16,17],

$$\Phi(z,s,v)=\sum_{m=0}^{\infty}\frac{z^m}{(m+v)^s} \tag{4}$$

$(v\in Z^+; Re(s)>1$ when $|z|=1,\ s\in C$ when $|z|<1).$

In [17–26], we can see a more cleared about the exposition of the different generalizations properties and applications of the Hurwitz–Lerch zeta functions. For example Garg et al. [21], Lin and Srivastava [27] and Goyal and Laddha [23], created certain remarkable extensions of the Hurwitz–Lerch zeta function $\Phi(z,s,v)$ given in (4), that are respectively shown by

$$\Phi_\delta^*(z,s,v)=\sum_{m=0}^{\infty}\frac{(\delta)_m}{m!}\frac{z^m}{(m+v)^s} \tag{5}$$

$(\delta \in C; v \in Z^+; Re(s-\delta) > 1$ when $|z| = 1$, $s \in C$ when $|z| < 1)$.

$$\Phi_{\delta,y}^{(\eta,w)}\left(z,s,v\right)=\sum_{m=0}^{\infty}\frac{\left(\varsigma\right)_{\eta m}}{\left(\gamma\right)_{wm}}\frac{z^m}{\left(m+v\right)^s} \tag{6}$$

$(\varsigma \in C; \gamma, v \in Z^+;\ \eta, w \in R^+; w > \eta$ when $w = \eta$ and $Re(s-\varsigma+\gamma) > 1$ when $|z| = 1$, $z \in C; w = \eta$ and $s \in C$ when $|z| < 1)$.
and

$$\Phi_{\delta,\varsigma;y}\left(z,s,v\right)=\sum_{m=0}^{\infty}\frac{\left(\delta\right)_m\left(\varsigma\right)_m}{\left(\gamma\right)_m\left(m+v\right)^s}\frac{z^m}{m!} \tag{7}$$

$(\varsigma, \delta \in C; \gamma, v \in Z^+;\ Re(s+\gamma-\delta-\varsigma) > 1$ when $|z| = 1, s \in C$ when $|z| < 1)$.

where $(\ell)_n$ for ($\ell \in C$) represent the Pochhammer symbol given by [28], the Pochhammer linked with the Gamma function given as:

$$(\ell)_n := \frac{\Gamma(\ell+n)}{\Gamma(\ell)},$$

where $(\ell)_0 = 1,\ (\ell)_n = \ell(\ell+1)(\ell+2)\ldots(\ell+n-1),\ \ n \in \mathbb{N}$.

The generalized Hurwitz–Lerch zeta function involving the extended beta function introduced by Parmar and Raina in 2014 [22], and [29] given by

$$\Phi_{\delta,\varsigma;y}\left(z,s,v;\rho\right)=\sum_{m=0}^{\infty}\frac{\left(\delta\right)_m}{m!}\frac{B_\rho\left(\varsigma+m,\gamma-\varsigma\right)}{B\left(\varsigma,\gamma-\varsigma\right)}\frac{z^m}{\left(m+v\right)^s} \tag{8}$$

$(\rho \ge 0; \varsigma, \delta \in C; \gamma, v \in Z^+;\ Re(s+\gamma-\delta-\varsigma) > 1$ when $|z| = 1, s \in C$ when $|z| < 1)$.

The following extension beta function introduced by Chaudhry et al. [29],

$$B(\hbar_1,\hbar_2;\rho)=\int_0^1 t^{\hbar_1-1}(1-t)^{\hbar_2-1}\exp\left(-\frac{\rho}{t\left(1-t\right)}\right)dt \tag{9}$$

Furthermore, Choi et al. [30] created the underlying generalization of extended beta functions given by:

$$B_{\rho,q}(\hbar_1,\hbar_2)=\int_0^1 t^{\hbar_1-1}(1-t)^{\hbar_2-1}\exp\left(-\frac{\rho}{t}-\frac{q}{1-t}\right)dt \tag{10}$$

$(Re(q) > 0; Re(\rho) > 0$ and $Re(\hbar_1) > 0; Re(\hbar_2) > 0)$.

Motivated by those different fascinating extensions of the Hurwitz-Lerch zeta function, researchers have created an extension of the generalized Hurwitz-Lerch zeta function that includes the extended beta function $B_{p,q}(\hbar_1,\hbar_2;\rho,q)$. In [30] a new extension of the generalized Hurwitz-Lerch zeta functions involving extended beta function (10) given by

$$\Phi_{\delta,\varsigma;y}\left(z,s,v;\rho,q\right)=\sum_{m=0}^{\infty}\frac{\left(\delta\right)_m}{m!}\frac{B_{\rho,q}\left(\varsigma+m,\gamma-\varsigma\right)}{B\left(\varsigma,\gamma-\varsigma\right)}\frac{z^m}{\left(m+v\right)^s} \tag{11}$$

$(q \ge 0, \rho \ge 0; \varsigma, \delta \in C; \gamma, v \in Z^+;\ Re(s+\gamma-\delta-\varsigma) > 1$ when $|z| = 1, s \in C$ when $|z| < 1)$.

Now define a normalized function of $\Phi_{\delta,\varsigma;y}\left(z,s,v;\rho,q\right)$ as:

$$\Phi_{\delta,\varsigma;y}\left(z,s,v;\rho,q\right)=z+\sum_{m=2}^{\infty}\frac{\Gamma\left(\delta+m\right)}{\delta!m!}\frac{B_{\rho,q}\left(\varsigma+m,\gamma-\varsigma\right)}{B_{\rho,q}\left(\varsigma+1,\gamma-\varsigma\right)}\left(\frac{1+v}{m+v}\right)^s z^m. \tag{12}$$

Analogous to the convolution tool, we consider the following new convolution operator:

$$F_{\delta,\varsigma;y}^{s,v;\rho,q} : \Gamma \to \Gamma$$

$$\begin{aligned} F_{\delta,\varsigma;y}^{s,v;\rho,q} g(z) &= \Phi_{\delta,\varsigma;y}(z,s,v;\rho,q) * g(z) \\ &= z + \sum_{m=2}^{\infty} \frac{\Gamma(\delta+m)}{\delta!m!} \frac{B_{\rho,q}(\varsigma+m,\gamma-\varsigma)}{B_{\rho,q}(\varsigma+1,\gamma-\varsigma)} \left(\frac{1+v}{m+v}\right)^s a_m z^m, \end{aligned} \tag{13}$$

The following differential operator introduced by Salagean's [31], for a function g in Γ:

$$D_\lambda^0 g(z) = g(z) \tag{14}$$

$$D_\lambda^1 g(z) = (1-\lambda) g(z) + \lambda z g'(z) = D_\lambda g(z), \qquad 0 \le \lambda \le 1, \tag{15}$$

$$D_\lambda^T g(z) = D_\lambda \left(D_\lambda^{T-1} g(z)\right), \qquad T \in N_0. \tag{16}$$

then

$$D_\lambda^T g(z) = z + \sum_{m=2}^{\infty} \left[1+(m-1)\lambda\right]^T a_m z^m. \tag{17}$$

Note that for $\lambda = 1$, Salagean's differential operator [31].

IMPOSED DIFFERENTIAL OPERATOR

By using a function $F_{\delta,\varsigma;y}^{s,v;\rho,q} g(z)$ given in (13), we introduce the following Salagean type differential operator: for a function $g \in \Gamma, T \in N_0$, and $H_\lambda^T : \Gamma \to \Gamma$ such that

$$H_\lambda^0 g(z) = g(z) \tag{18}$$

$$\begin{aligned} H_\lambda^1 g(z) &= (1-\lambda) F_{\delta,\varsigma;y}^{s,v;\rho,q} g(z) + \lambda z \left(F_{\delta,\varsigma;y}^{s,v;\rho,q} g(z)\right)', \qquad 0 \le \lambda \le 1, \\ &= z + \sum_{m=2}^{\infty} \frac{\Gamma(\delta+m)}{\delta!m!} \frac{B_{\rho,q}(\varsigma+m,\gamma-\varsigma)}{B_{\rho,q}(\varsigma+1,\gamma-\varsigma)} \left(\frac{1+v}{m+v}\right)^s \left(1+(m-1)\lambda\right) a_m z^m \end{aligned} \tag{19}$$

then

$$\begin{aligned} H_\lambda^T g(z) &= H_\lambda^1 \left(H_\lambda^{T-1} g(z)\right) \\ &= z + \sum_{m=2}^{\infty} \left[\frac{\Gamma(\delta+m)}{\delta!m!} \frac{B_{\rho,q}(\varsigma+m,\gamma-\varsigma)}{B_{\rho,q}(\varsigma+1,\gamma-\varsigma)} \left(\frac{1+v}{m+v}\right)^s \left(1+(m-1)\lambda\right)\right]^T a_m z^m \end{aligned} \tag{20}$$

$(q \ge 0,\ \rho \ge 0;\ 0 \le \lambda \le 1;\ \ \varsigma,\delta \in C;\ \gamma, v \in Z^+;\ Re(s+\gamma-\delta-\varsigma) > 1$ when $|z| = 1,\ s \in C$ when $|z| < 1,$ $T \in N_0)$.

For convenience

$$Q(\delta,v,\lambda,T) = \left[\frac{\Gamma(\delta+m)}{\delta!m!} \frac{B_{\rho,q}(\varsigma+m,\gamma-\varsigma)}{B_{\rho,q}(\varsigma+1,\gamma-\varsigma)} \left(\frac{1+v}{m+v}\right)^s \left(1+(m-1)\lambda\right)\right]^T$$

$$H_\lambda^T g(z) = z + \sum_{m=2}^{\infty} Q(\delta,v,\lambda,T)\, a_m z^m. \tag{21}$$

For suitable choices of the parameters involved, the above-defined operator $H_\lambda^T g(z)$ offers different other Salagean's differential operator type that are shown in earlier works. For instance:

1. For $T=0,$ the operator coincides to $g(z).$
2. For $T=1,\ \lambda=\rho=q=0,\ \varsigma=\delta=v=1,\ \gamma=2,$ and $s=-1$ the operator corresponds to $g(z).$
3. For $T=1,\ \rho=q=0,\ \varsigma=\delta=v=1,\ \gamma=2,$ and $s=-1$ the operator corresponds to Salagean's differential operator [31].

Using differential operator since by (2.4), we define and investigate $\kappa-S_p(\alpha)$ consisting of functions $g\in\Gamma$ that achieve the conditions:

$$Re\left\{\frac{z\left(H_\lambda^T g(z)\right)'}{H_\lambda^T g(z)}\right\}\geq\kappa\left|\frac{z\left(H_\lambda^T g(z)\right)'}{H_\lambda^T g(z)}-1\right|+\alpha, \tag{22}$$

where $0\leq\alpha<1$ and $0\leq\kappa<\infty$.

Let G denote the subclass of Γ consisting of functions g in Δ of the form

$$g(z)=z-\sum_{m=2}^{\infty}a_m z^m, \qquad \text{for } (a_m\geq 0 \text{ and } z\in\Delta). \tag{23}$$

Also let

$$\kappa-GS_p(\alpha,\lambda,\gamma)=S_p(\alpha,\lambda,\gamma)\cap G.$$

COEFFICIENT BOUNDS

In this section, several basic geometric properties of the uniformly star-like class $\kappa-GS_p(\alpha,\lambda,\gamma)$ are investigated and presented.

Theorem 3.1. A function g defined by (23), is in $g(z)\in\kappa-GS_p(\alpha,\lambda,\gamma)$ if and only if

$$\sum_{m=2}^{\infty}\big(m(\kappa+1)-(\kappa+\alpha)\big)Q\left(\delta,v,\lambda,T\right)a_m\leq 1-\alpha, \qquad (0\leq\alpha<1) \tag{24}$$

Proof. We suppose that the inequality (3.1) holds right and let $|z|=1$. It suffices to show that

$$Re\left\{\frac{z\left(H_\lambda^T g(z)\right)'}{H_\lambda^T g(z)}\right\}\geq\kappa\left|\frac{z\left(H_\lambda^T g(z)\right)'}{H_\lambda^T g(z)}-1\right|+\alpha,$$

we obtain,

$$\kappa\left|\frac{z\left(H_\lambda^T g(z)\right)'}{H_\lambda^T g(z)}-1\right|-Re\left\{\frac{z\left(H_\lambda^T g(z)\right)'}{H_\lambda^T g(z)}-\alpha\right\}$$

$$\leq(\kappa+1)\left|\frac{z\left(H_\lambda^T g(z)\right)'}{H_\lambda^T g(z)}-1\right|\leq\frac{(\kappa+1)\sum_{m=2}^{\infty}\left(m-1\right)Q\left(\delta,v,\lambda,T\right)a_m}{1-\sum_{m=2}^{\infty}Q\left(\delta,v,\lambda,T\right)a_m}.$$

The last expression is bounded above by $(1-\alpha)$ if

$$\sum_{m=2}^{\infty}\left(m(\kappa+1)-(\kappa+\alpha)\right)Q\left(\delta,v,\lambda,T\right)a_m\le 1-\alpha,$$

which is right by hypothesis. Hence we have $g(z)\in\kappa-GS_p(\alpha,\lambda,\gamma).$

To prove converse, suppose that $g(z)$ given by (23) in $\kappa-GS_p(\alpha,\lambda,\gamma).$ So that choosing the value of z on the positive real axis, the inequality (22) readily gains

$$\frac{1-\sum_{m=2}^{\infty}mQ\left(\delta,v,\lambda,T\right)\,a_m\,z^{m-1}}{1-\sum_{m=2}^{\infty}Q\left(\delta,v,\lambda,T\right)\,a_m\,z^{m-1}}-\alpha\ge\kappa\left|\frac{\sum_{m=2}^{\infty}(m-1)Q\left(\delta,v,\lambda,T\right)\,a_m\,z^{m-1}}{1-\sum_{m=2}^{\infty}Q\left(\delta,v,\lambda,T\right)\,a_m\,z^{m-1}}\right|,$$

letting $z\to 1^-$ along the real axis, we yield,

$$\frac{1-\sum_{m=2}^{\infty}mQ\left(\delta,v,\lambda,T\right)\,a_m}{1-\sum_{m=2}^{\infty}Q\left(\delta,v,\lambda,T\right)\,a_m}-\alpha\ge\kappa\frac{\sum_{m=2}^{\infty}(m-1)Q\left(\delta,v,\lambda,T\right)\,a_m}{1-\sum_{m=2}^{\infty}Q\left(\delta,v,\lambda,T\right)\,a_m},$$

then

$$\frac{(1-\alpha)-\sum_{m=2}^{\infty}\left(m\left(\kappa+1\right)-\left(\kappa+\alpha\right)\right)Q\left(\delta,v,\lambda,T\right)\,a_m}{1-\sum_{m=2}^{\infty}Q\left(\delta,v,\lambda,T\right)\,a_m}\ge 0,$$

Therefore, we have assertion (24). Finally equality holds for the function g defined by

$$g(z)=z-\frac{(1-\alpha)}{\left(m(\kappa+1)-(\kappa+\alpha)\right)Q\left(\delta,v,\lambda,T\right)}z^m,\qquad (m\ge 2).$$

Corollary 3.1 Let the function $g(z)$ be defined by (23) if $g(z)\in\kappa-GS_p(\alpha,\lambda,\gamma)$
then

$$\sum_{m=2}^{\infty}\left(m(\kappa+1)-(\kappa+\alpha)\right)Q\left(\delta,v,\lambda,T\right)a_m\le 1-\alpha$$

$$\left(m(\kappa+1)-(\kappa+\alpha)\right)Q\left(\delta,v,\lambda,T\right)a_m\le\sum_{m=2}^{\infty}\left(m(\kappa+1)-(\kappa+\alpha)\right)Q\left(\delta,v,\lambda,T\right)a_m\le 1-\alpha$$

$$\left(m(\kappa+1)-(\kappa+\alpha)\right)Q\left(\delta,v,\lambda,T\right)a_m\le 1-\alpha,\qquad (m\ge 2),$$

$$a_m\le\frac{(1-\alpha)}{\left(m(\kappa+1)-(\kappa+\alpha)\right)Q\left(\delta,v,\lambda,T\right)},\qquad (m\ge 2).\qquad (25)$$

The result is sharp for the function

$$g(z)=z-\frac{(1-\alpha)}{\left(m(\kappa+1)-(\kappa+\alpha)\right)Q\left(\delta,v,\lambda,T\right)}z^m,\qquad (m\ge 2).\qquad (26)$$

GROWTH THEOREMS

Theorem 4.1 Let the function $g(z)$ defined by (23) be in the class $\kappa - GS_p(\alpha,\lambda,\gamma)$, then

$$\left|g(z)\right| \ge |z| - \frac{(1-\alpha)}{(\kappa-\alpha+2)}\left[\frac{(\delta+1)}{2}\frac{B_{\rho,q}(\varsigma+2,\gamma-\varsigma)}{B_{\rho,q}(\varsigma+1,\gamma-\varsigma)}\left(\frac{1+v}{2+v}\right)^s(1+\lambda)\right]^{-T}|z|^2$$
$$\le |z| + \frac{(1-\alpha)}{(\kappa-\alpha+2)}\left[\frac{(\delta+1)}{2}\frac{B_{\rho,q}(\varsigma+2,\gamma-\varsigma)}{B_{\rho,q}(\varsigma+1,\gamma-\varsigma)}\left(\frac{1+v}{2+v}\right)^s(1+\lambda)\right]^{-T}|z|^2 \qquad (27)$$

with equality for

$$\left|g(z)\right| = |z| - \frac{(1-\alpha)}{(\kappa-\alpha+2)}\left[\frac{(\delta+1)}{2}\frac{B_{\rho,q}(\varsigma+2,\gamma-\varsigma)}{B_{\rho,q}(\varsigma+1,\gamma-\varsigma)}\left(\frac{1+v}{2+v}\right)^s(1+\lambda)\right]^{-T}|z|^2 \;,\; (z=\pm r). \qquad (28)$$

Proof. Let $g(z)\in\kappa - GS_p(\alpha,\lambda,\gamma)$. In view of Theorem 3.1,

$$\Psi(m) = \left(m(\kappa-1)-(\kappa+\alpha)\right)Q\left(\delta,v,\lambda,T\right),$$

we have

$$\Psi(2)\sum_{m=2}^{\infty}a_m \le \sum_{m=2}^{\infty}\Psi(m)a_m \le 1-\alpha,$$

that is

$$\sum_{m=2}^{\infty}a_m \le \frac{1-\alpha}{\Psi(2)}.$$

Thus, we have

$$\left|g(z)\right| \ge |z| - \sum_{m=2}^{\infty}a_m\left|z^m\right| \ge |z| - |z|^2\sum_{m=2}^{\infty}a_m$$
$$\ge |z| - \frac{(1-\alpha)}{(\kappa-\alpha+2)}\left[\frac{(\delta+1)}{2}\frac{B_{\rho,q}(\varsigma+2,\gamma-\varsigma)}{B_{\rho,q}(\varsigma+1,\gamma-\varsigma)}\left(\frac{1+v}{2+v}\right)^s(1+\lambda)\right]^{-T}|z|^2.$$

Similarly, we get

$$\left|g(z)\right| \le |z| + |z|^2\sum_{m=2}^{\infty}a_m,$$

$$\left|g(z)\right| \le |z| + \frac{(1-\alpha)}{(\kappa-\alpha+2)}\left[\frac{(\delta+1)}{2}\frac{B_{\rho,q}(\varsigma+2,\gamma-\varsigma)}{B_{\rho,q}(\varsigma+1,\gamma-\varsigma)}\left(\frac{1+v}{2+v}\right)^s(1+\lambda)\right]^{-T}|z|^2.$$

Finally, the result is sharp for the function

$$\left|g(z)\right| = |z| - \frac{(1-\alpha)}{(\kappa-\alpha+2)}\left[\frac{(\delta+1)}{2}\frac{B_{\rho,q}(\varsigma+2,\gamma-\varsigma)}{B_{\rho,q}(\varsigma+1,\gamma-\varsigma)}\left(\frac{1+v}{2+v}\right)^s(1+\lambda)\right]^{-T}|z|^2 \;,\; (z=\pm r).$$

Theorem 4.2 Let the function $g(z)$ defined by (23) be in the class $\kappa - GS_p(\alpha,\lambda,\gamma)$, then

$$|g'(z)| \geq 1 - \frac{2(1-\alpha)}{(\kappa-\alpha+2)}\left[\frac{(\delta+1)}{2}\frac{B_{\rho,q}(\varsigma+2,\gamma-\varsigma)}{B_{\rho,q}(\varsigma+1,\gamma-\varsigma)}\left(\frac{1+v}{2+v}\right)^{s}(1+\lambda)\right]^{-T}|z|$$
$$\leq 1 + \frac{2(1-\alpha)}{(\kappa-\alpha+2)}\left[\frac{(\delta+1)}{2}\frac{B_{\rho,q}(\varsigma+2,\gamma-\varsigma)}{B_{\rho,q}(\varsigma+1,\gamma-\varsigma)}\left(\frac{1+v}{2+v}\right)^{s}(1+\lambda)\right]^{-T}|z| \tag{29}$$

with equality for

$$|g'(z)| = 1 - \frac{2(1-\alpha)}{(\kappa-\alpha+2)}\left[\frac{(\delta+1)}{2}\frac{B_{\rho,q}(\varsigma+2,\gamma-\varsigma)}{B_{\rho,q}(\varsigma+1,\gamma-\varsigma)}\left(\frac{1+v}{2+v}\right)^{s}(1+\lambda)\right]^{-T}|z|\ ,\ (z=\pm r). \tag{30}$$

Proof. Let $g(z) \in \kappa - GS_p(\alpha,\lambda,\gamma)$. In view of Theorem 3.1,

$$\Psi(m) = \left(m(\kappa+1) - (\kappa+\alpha)\right)Q(\delta,v,\lambda,T),$$

we have

$$\frac{\Psi(2)}{2}\sum_{m=2}^{\infty} m a_m \leq \sum_{m=2}^{\infty}\frac{\Psi(m)}{m} m a_m = \sum_{m=2}^{\infty}\Psi(m)a_m \leq 1-\alpha,$$

that is

$$\sum_{m=2}^{\infty} m a_m \leq \frac{2(1-\alpha)}{\Psi(2)}.$$

Thus we have

$$|g'(z)| \geq 1 - \sum_{m=2}^{\infty} a_m \left|z^{m-1}\right| \geq 1 - |z|\sum_{m=2}^{\infty} a_m$$
$$\geq 1 - \frac{2(1-\alpha)}{(\kappa-\alpha+2)}\left[\frac{(\delta+1)}{2}\frac{B_{\rho,q}(\varsigma+2,\gamma-\varsigma)}{B_{\rho,q}(\varsigma+1,\gamma-\varsigma)}\left(\frac{1+v}{2+v}\right)^{s}(1+\lambda)\right]^{-T}|z|.$$

Similarly, we get

$$|g'(z)| \leq 1 + |z|\sum_{m=2}^{\infty} a_m,$$

$$|g'(z)| \leq 1 + \frac{2(1-\alpha)}{(\kappa-\alpha+2)}\left[\frac{(\delta+1)}{2}\frac{B_{\rho,q}(\varsigma+2,\gamma-\varsigma)}{B_{\rho,q}(\varsigma+1,\gamma-\varsigma)}\left(\frac{1+v}{2+v}\right)^{s}(1+\lambda)\right]^{-T}|z|.$$

The result is sharp for the function

$$|g'(z)| = 1 - \frac{2(1-\alpha)}{(\kappa-\alpha+2)}\left[\frac{(\delta+1)}{2}\frac{B_{\rho,q}(\varsigma+2,\gamma-\varsigma)}{B_{\rho,q}(\varsigma+1,\gamma-\varsigma)}\left(\frac{1+v}{2+v}\right)^{s}(1+\lambda)\right]^{-T}|z|\ ,\ (z=\pm r).$$

RADII OF CONVEXITY, STAR-LIKENESS, AND CLOSE-TO-CONVEXITY

In this section, radii of Convexity, star-likeness and close-to-convexity for functions belonging to the class $\kappa - GS_p(\alpha,\lambda,\gamma)$ are obtained.

Theorem 5.1 Let the function $g(z)$ defined by (2.6) be in the class $\kappa - GS_p(\alpha,\lambda,\gamma)$, then

i. $g(z)$ is convex of order $\beta(0 \leq \beta < 1)$ in $|z| < r_1$, where

$$r_1 = \inf_{m\ge 2}\left\{\left[\frac{(1-\beta)}{m(m-\beta)}\frac{\left(m(\kappa+1)-(\kappa+\alpha)\right)Q(\delta,v,\lambda,T)}{(1-\alpha)}\right]^{\frac{1}{m-1}}\right\}. \tag{31}$$

ii. $g(z)$ is star-like of order $\beta(0\le\beta<1)$ in $|z|<r_2$, where

$$r_2 = \inf_{m\ge 2}\left\{\left[\frac{(1-\beta)}{(m-\beta)}\frac{\left(m(\kappa+1)-(\kappa+\alpha)\right)Q(\delta,v,\lambda,T)}{(1-\alpha)}\right]^{\frac{1}{m-1}}\right\}. \tag{32}$$

iii. $g(z)$ is closed-to-convex of order $\beta(0\le\beta<1)$ in $|z|<r_3$, where

$$r_3 = \inf_{m\ge 2}\left\{\left[\frac{(1-\beta)}{m}\frac{\left(m(\kappa+1)-(\kappa+\alpha)\right)Q(\delta,v,\lambda,T)}{(1-\alpha)}\right]^{\frac{1}{m-1}}\right\}. \tag{33}$$

All of these results is sharp for the function $g(z)$ given by (20).

Proof.(i) It is enough to show that

$$\left|\frac{zg''(z)}{g'(z)}\right|\le 1-\beta, \qquad \text{for } |z|<r_1, \text{ and } (0\le\beta<1).$$

where r_1 is given by (5.2). Indeed, we find from (2.6) that

$$\left|\frac{z\left(H_\lambda^T g(z)\right)''}{\left(H_\lambda^T g(z)\right)'}\right| \le \frac{\sum_{m=2}^{\infty} m(m-1)a_m|z|^{m-1}}{1-\sum_{m=2}^{\infty} ma_m|z|^{m-1}}.$$

Thus, we have

$$\left|\frac{zg''(z)}{g'(z)}\right|\le 1-\beta,$$

if and only if

$$\sum_{m=2}^{\infty} m(m-\beta)a_m|z|^{m-1}\le 1-\beta. \tag{34}$$

But, by Theorem 3.1,(34) will be right if

$$m(m-\beta)|z|^{m-1}\le(1-\beta)\frac{\left(m(\kappa+1)-(\kappa+\alpha)\right)Q(\delta,v,\lambda,T)}{(1-\alpha)}$$

that is, if

$$|z|\le\left[\frac{(1-\beta)}{m(m-\beta)}\frac{\left(m(\kappa+1)-(\kappa+\alpha)\right)Q(\delta,v,\lambda,T)}{(1-\alpha)}\right]^{\frac{1}{m-1}}, \qquad (m\ge 2).$$

Or

$$r_1 = \inf_{m\geq 2}\left\{\left[\frac{(1-\beta)}{m(m-\beta)}\frac{\left(m(\kappa+1)-(\kappa+\alpha)\right)Q(\delta,v,\lambda,T)}{(1-\alpha)}\right]^{\frac{1}{m-1}}\right\}.$$

(ii) It is enough to show that

$$\left|\frac{zg'(z)}{g(z)}-1\right|\leq 1-\beta, \qquad \text{for } |z|<r_2 \text{ and } (0\leq\beta<1).$$

where r_2 is given by (31). Indeed, we find from (2.6) that

$$\left|\frac{z\left(H_\lambda^T g(z)\right)'}{H_\lambda^T g(z)}-1\right|\leq\frac{\sum_{m=2}^{\infty}(m-1)a_m|z|^{m-1}}{1-\sum_{m=2}^{\infty}a_m|z|^{m-1}}.$$

Thus, we have

$$\left|\frac{zg'(z)}{g(z)}-1\right|\leq 1-\beta,$$

if and only if

$$\sum_{m=2}^{\infty}(m-\beta)a_m|z|^{m-1}\leq 1-\beta. \tag{35}$$

But, by Theorem 3.1,(33) will be right if

$$(m-\beta)|z|^{m-1}\leq(1-\beta)\frac{\left(m(\kappa+1)-(\kappa+\alpha)\right)Q(\delta,v,\lambda,T)}{(1-\alpha)}$$

that is, if

$$|z|\leq\left[\frac{(1-\beta)}{(m-\beta)}\frac{\left(m(\kappa+1)-(\kappa+\alpha)\right)Q(\delta,v,\lambda,T)}{(1-\alpha)}\right]^{\frac{1}{m-1}}, \qquad (m\geq 2).$$

Or

$$r_2 = \inf_{m\geq 2}\left\{\left[\frac{(1-\beta)}{(m-\beta)}\frac{\left(m(\kappa+1)-(\kappa+\alpha)\right)Q(\delta,v,\lambda,T)}{(1-\alpha)}\right]^{\frac{1}{m-1}}\right\}.$$

(iii) It is enough to show that

$$|g'(z)-1|\leq 1-\beta, \qquad \text{for } |z|<r_3, \text{ and } (0\leq\beta<1).$$

where r_3 is given by (33). Indeed, we find from (23) that

$$\left|\left(H_\lambda^T g(z)\right)'-1\right|=\sum_{m=2}^{\infty}ma_m|z|^{m-1}.$$

Thus, we have

$$|g'(z)-1|\leq 1-\beta,$$

if and only if

$$\sum_{m=2}^{\infty}ma_m|z|^{m-1}\leq 1-\beta. \tag{36}$$

But, by Theorem 3.1,(35) will be right if

$$m|z|^{m-1} \le (1-\beta)\frac{\left(m(\kappa+1)-(\kappa+\alpha)\right)Q(\delta,v,\lambda,T)}{(1-\alpha)}$$

that is, if

$$|z| \le \left[\frac{(1-\beta)}{m}\frac{\left(m(\kappa+1)-(\kappa+\alpha)\right)Q(\delta,v,\lambda,T)}{(1-\alpha)}\right]^{\frac{1}{m-1}}, \qquad (m \ge 2).$$

Or

$$r_3 = \inf_{m\ge 2}\left\{\left[\frac{(1-\beta)}{m}\frac{\left(m(\kappa+1)-(\kappa+\alpha)\right)Q(\delta,v,\lambda,T)}{(1-\alpha)}\right]^{\frac{1}{m-1}}\right\}.$$

EXTREME POINTS

Theorem 6.1 Let

$$g_1(z) = z,$$

$$g_m(z) = z - \frac{(1-\alpha)}{\left(m(\kappa+1)-(\kappa+\alpha)\right)Q(\delta,\gamma,v,T)} z^m, \qquad (m \ge 2).$$

Then $g(z) \in \kappa - GS_p(\alpha,\lambda,\gamma)$ if and only if it can be expressed in the following form:

$$g(z) = \sum_{m=1}^{\infty} \mu_m g_m(z),$$

where

$$\mu_m \ge 0, \qquad \sum_{m=1}^{\infty} \mu_m = 1.$$

Proof. We suppose that

$$g(z) = \sum_{m=1}^{\infty} \mu_m g_m(z),$$

$$= z - \sum_{m=2}^{\infty} \mu_m \frac{(1-\alpha)}{\left(m(\kappa+1)-(\kappa+\alpha)\right)Q(\delta,v,\lambda,T)} z^m.$$

Then, form Theorem 3.1, we have

$$\sum_{m=2}^{\infty}\left(\left(m(\kappa+1)-(\kappa+\alpha)\right)Q(\delta,v,\lambda,T)\frac{(1-\alpha)}{\left(m(\kappa+1)-(\kappa+\alpha)\right)Q(\delta,v,\lambda,T)}\mu_m\right)$$

$$= (1-\alpha)\sum_{m=2}^{\infty}\mu_m = (1-\alpha)(1-\mu_1) \le 1-\alpha.$$

So, looked at Theorem 3.1, we find that $g(z) \in \kappa - GS_p(\alpha,\lambda,\gamma)$

Conversely, we suppose that $g(z) \in \kappa - GS_p(\alpha,\lambda,\gamma)$, since

$$a_m \leq \frac{(1-\alpha)}{\big(m(\kappa+1)-(\kappa+\alpha)\big)Q(\delta,v,\lambda,T)}, \qquad (m \geq 2).$$

set

$$\mu_m = \frac{\big(m(\kappa+1)-(\kappa+\alpha)\big)Q(\delta,v,\lambda,T)}{(1-\alpha)} a_m, \qquad (m \geq 2),$$

$$\mu_1 = 1 - \sum_{m=2}^{\infty} \mu_m.$$

Thus clearly, we have

$$g(z) = \sum_{m=1}^{\infty} \mu_m g_m(z).$$

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