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Connected Fibrewise Topological Spaces

N F Mohammed and Y Y Yousif

Department of Mathematics, College of Education for Pure Sciences/ Ibn-AL-Haithem, University of Baghdad, Iraq

Email: nadia79math@yahoo.com

Abstract. Fibrewise topological spaces theory is a relatively new branch of mathematics, less than three decades old, arisen from algebraic topology. It is a highly useful tool and played a pivotal role in homotopy theory. Fibrewise topological spaces theory has a broad range of applications in many sorts of mathematical study such as Lie groups, differential geometry and dynamical systems theory. Moreover, one of the main objects, which is considered in fibrewise topological spaces theory is connectedness. In this regard, we of the present study introduce the concept of connected fibrewise topological spaces and study their main results.

1- Introduction

Fibrewise topological space is today one of most rapidly expanding areas of mathematical thought. A fertile ground for the growth of fibrewise topological spaces is provided by general topology and algebraic topology. Although it has numerous applications in different fields of mathematics like differential geometry and much else. The magnificent definition of fibreweise compact-open topology with fibrewise mapping space was given by Booth and Brown [1, 2] in 1978. In 1982, Niefield [3] confined that the fibrewise space and base space are uniforms. Furthermore, the condition of Niefield was weakened to the usual topological space by James [4] in 1985. A new era in theory of fibrewise has been developing due to work of James [5] in 1989, in his book entitled "Fibrewise Topology", which summarized the fundamental results of the past. Moreover, the book titled "Fibrewise Homotopy Theory" was written by James and Crabb [6] in 1998, that further enhanced the fibrewise topology space theory. In literature [7, 8], Yousif studied fibrewise weakly continuous functions, and introduced some result on fibrewise Lindelof and locally Lindelof topological spaces. Recently fibrewise topological spaces theory has been attracted and widely used by the group of researchers like Hussain [9] and Ghafil [10].
On the other hand, one of the main object considered ingeneral topology is connectedness and it has been studied by many investigators like Mohammed [11], Dasgupta and Chakrabarti [12], and so on. However, the connectedness of fibrewise topology spaces, in general, is still an open problem. For this reason, our interest in this paper is to introduce the concept of connected fibrewise topological spaces and study their properties.

The outline of the paper is as follows. In Section 1, a brief introduction is given. Some main definitions are provided in Section 2. The main results of the paper are in Section 3, where the concept of connected fibrewise topological spaces is introduced. Furthermore, we prove several propositions concerning with this concept.

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2- Preliminaries

In this section, we review some relevant definitions and set up the terminology, which are provided in James [5], to be used throughout the present study.

Let B be a base set, then a fibrewise set over B is a set X equipped with a projection map $p : X \to B$. For any $b \in B$, the subset $X_b = p^{-1}(b)$ of X is called the fibre over b. It is worthy of note that the subspace of X is a fibrewise space over B, when the map p is restricted.

Definition 2.1. A map θ between two fibrewise sets X, with projection p, and Y, with projection q, over B is known as fibrewise if $q \circ \theta = p$.

It has to be noted from Definition 2.1 that the set $\theta(X_b)$ is a subset of Y_b, $\forall b \in B$.

Definition 2.2. Let X be a fibrewise set over B such that B is a topological space. The topology on X is called fibrewise topology if the projection map p is continuous.

Definition 2.3. A fibrewise function θ between two fibrewise topological spaces X and Y over B is said to be continuous if, for each $x \in X_b$ such that $b \in B$, $\theta^{-1}(V)$ is an open set of x for each open set V of $\theta(x)$ in Y.

In the sequel of this paper, the closure, the interior and the boundary of a set A are denoted by c (A), i (A) and $b(A)$, respectively.

3- Connected Fibrewise Topological Spaces

Connectedness is one of the main topological notions and very exciting research that takes place in general topology. In this regards, the present section concentrates on defining the concept of connectedness on fibrewise topology spaces and studying their properties. Now, we begin with the following definition.

Definition 3.1. A fibrewise topological space X over B is called connected fibrewise topological space if for each two points x and y in X_b, where that $b \in B$, X cannot be represented by the union of two disjoint nonempty, open subsets of X, say U and V, where $x \in U$ and $y \in V$. Otherwise, X is said to be disconnected fibrewise topological space.

To develop a better understanding about this definition, we provide the following example.

Example 3.1. Consider $X = \{a, b, c, d\}$ with topology $\tau_X = \{X, \emptyset, \{a\}, \{c\}$, {a,c}} on X, and $B = X$ with $\tau_B = \{B, \emptyset, \{a\}\}$. Define a projection function $p : X \to B$ by $p(a) = p(c) = a$ and $p(b) = p(d) = b$.

Observe that the projection function is continuous since the inverse of each member of τ_B on B is a member of τ_X on X. On other hand, the space X is a connected fibrewise topological space since for each two points x and y in X_b, where $b \in B$, there is no two disjoint open, nonempty subsets of X, say U and
V, where $x \in U$, $y \in V$, and $X = U \cup V$.

The next theorem illustrates that fibrewise connectedness is a property which is preserved by fibrewise continuous function.

Theorem 3.1. Suppose θ is a surjective fibrewise continuous function from a connected fibrewise topological space X onto a fibrewise topological space Y. Then Y is connected fibrewise topological space.

Proof. Let x_1 and x_2 be two points in X_b, where $b \in B$. Since θ is a fibrewise function, there exist two points y_1 and y_1 in Y_b such that $y_1 = (x_1)$ and $y_2 = (x_2)$. By way of contradiction, assume Y be a disconnected fibrewise topological space. Consequently, Y can be represented by the union of two proper disjoint and open subsets in Y, say U and V, where $y_1 \in U$, $y_2 \in V$.

Hence, $x_1 \in \theta^{-1}(U)$ and $x_2 \in \theta^{-1}(V)$ because of θ is a fibrewise continuous function. Moreover, $\theta^{-1}(U)$, $\theta^{-1}(V)$ are two disjoint open, nonempty subsets in X. As a result, $X = \theta^{-1}(U) \cup \theta^{-1}(V)$ is a disconnected fibrewise topological space, which is a contradiction. Therefore, Y is a connected fibrewise topological space.

The following theorem gives some criteria for the connectedness of fibrewise topology spaces.

Theorem 3.2. Suppose X is any connected fibrewise topological space. Then the following statements are equivalent.

(1) X is a connected fibrewise topological space.
(2) X cannot represented by the union of two disjoint nonempty, closed subsets in X, say F_1 and F_2, where $x \in F_1$, $y \in F_1$; $\forall x, y \in X_b$.
(3) There is no clopen subset in X, say A, where $x \in A$ and $y \in A^c$, for each x and $y \in X_b$, $b \in B$.
(4) If A be any proper subset of X such that $x \in A$ and $y \in A^c$; $\forall x, y \in X_b$, then the boundary of A is nonempty set.

Proof. Statement (**1**) implies statement (**2**). Suppose $X = F_1 \cup F_2$, where F_1 and F_2 are disjoint closed, nonempty subsets of X and $x \in F_1$, $y \in F_2$, $\forall x, y \in X_b$, $b \in B$. However, $F_1 = F_2^c$ and $F_2 = F_1^c$ are also open subsets in X. Hence X is not connected fibrewise, which is a contradiction.

Statement (**2**) implies statement (**3**). Suppose A is a proper and clopen subset of X, where $x \in A$ and $y \in A^c, \forall x, y \in X_b, b \in B$. However, A is neither X nor $\emptyset$. Then A^c is also proper clopen subset of X. Thus $X = A \cup A^c$, contradicting (2).

Statement (**3**) implies statement (**4**). If A is a proper subset in X such that $x \in A$ and $y \in A^c$, $x, y \in X_b$, $b \in B$ and $b(A) = \emptyset$, then $c\,(A) = \dot{\imath}\imath\,(A) \cup b(A)$, we have $c\,(A) = \dot{\imath}\imath\,(A)$. On the other hand, $\dot{\imath}\imath\,(A) \subset A$ and $A \subset c\,(A)$, and hence $A = \dot{\imath}\imath\,(A) = c\,(A)$. Thus A is clopen subset, which is
a contradiction.

Statement (**4**) implies statement (**1**). Suppose X is represented by the union of two disjoint open, nonempty subsets of X, say, U and V, where $x \in U$, $y \in V$, $\forall x, y \in X_b, b \in B$. Then U and V are also closed. Therefore, $A = \dot{\imath}\imath\,(A) = c\,(A)$. However, $b(A) = c\,(A) - \dot{\imath}\imath\,(A)$; hence $b(A) = \emptyset$, a contradiction of (4).

Proposition 3.1. The fibrewise topological space X is connected if and only if there is no fibrewise continuous function from X onto Y, where $Y = \{0,1\}$ and has a discrete topology.

Proof. ⟹) Suppose there is a fibrewise continuous function from X onto Y. Then since X is a connected fibrewise, Y must be also (by Theorem 3.1), which is not the case.

⟸) Suppose X is a disconnected fibrewise. Thus, X can be written as the union of two disjoint open, nonempty subsets of X, say U and V , where $x \in U$, $y \in V$, $\forall x, y \in X_b$, $b \in B$. Define $\theta: X \rightarrow Y$ by $\theta(z) = 0$, if $z \in U$, and $\theta(z) = 1$, if $z \in V$. Then $x \in \theta^{-1}(\{0\}) = U$ and $y \in \theta^{-1}(\{1\}) = V$. Hence θ is
a fibrewise continuous function, which is a contradiction.

4 - Connectedness of Fibrewise Subspaces

In this section we define connected fibrewise subspace and investigate criteria for determining if a subspace is connected fibrewise.

Definition 4.1. Let X be a fibrewise topological space. The fibrewise subspace A in X is called connected fibrewise topological subspace if for each two points x and y in X_b, where $b \in B$, A can not represented by the union of two disjoint open, nonempty subsets of A, say U_1 and U_2, such that $x \in U_1$ and $y \in U_2$.
Otherwise, A is said to be disconnected fibrewise topological subspace.

Arbitrary fibrewise subspaces of connected fibrewise spaces need not be connected fibrewise as shown in the following example. Accordingly, the connectedness of fibrewise is not a hereditary property.

Example 4.1. Let (X, τ_X), (B, τ_B), and p be a connected fibrewise topological space, fibrewise topological base space and projection function, respectively, as in Example 3.1. Now let $Y = \{a, c\}$ with topology $\tau_Y = \{Y, \emptyset, \{a\}, \{c\}, \{a,c\}\}$ be a Fibrewise topological subspace of X. We notice that Y is disconnected fibrewise subspace of X since there exist two points a and c in X_a, where $a \in B$, and two disjoint open, nonempty subsets of A, $\{a\}$ and $\{c\}$, such that $a \in \{a\}$, $c \in \{c\}$ and $Y = \{a\} \cup \{c\}$. However, X is connected fibrewise space. Thus the connectedness of fibrewise is not a hereditary property.

Proposition 4.1. Suppose X is a fibrewise topological space and can be represented by the union of two disjoint nonempty, open subsets of X, say U and V. Let A be a connected fibrewise topological subspace of X and $x \in U \cap A, y \in V \cap A$, for each two points x and y in X_b, where $b \in B$. Then either $A \subset U$ or $A \subset V$.

Proof. If $U \cap A \neq \emptyset, V \cap A \neq \emptyset$ then $U \cap A$ and $V \cap A$ are nonempty, disjoint subsets of A which are open in A. However,

$$A = (U \cap A) \cap (V \cap A);$$

Thus A is not connected fibrewise. Therefore, either $U \cap A = \emptyset$, and hence $A \subset V$ or $V \cap A = \emptyset$ and hence $A \subset U$.

Definition 4.2. Let X be a fibrewise topological space. The two nonempty subsets S and T of X are called fibrewise-separated sets if and only if for each two points x and y in X_b, where $b \in B$, $x \in S, y \in T$, and $S \cap$ cl(T) = $c\,(S) \cap T = \emptyset$.

The following proposition gives a criterion for determining whether or not a subspace of a given space is connected fibrewise.

Proposition 4.2. Let X be a fibrewise space. Then a fibrewise subspace A of X is connected fibrewise if and only if A cannot expressed as the union of two fibrewise-separated sets.

Proof. $\Longrightarrow$) If A is disconnected fibrewise, then A can be represented by the union of two disjoint, nonempty subsets of A, say S and T, which are clopen in A. Moreover, $x \in S$, $y \in T$, $S \cap c\,(T) = \emptyset$ and $c\,(S) \cap T = \emptyset$, for each two points x and y in X_b, where $b \in B$. Suppose

$$x \in S \cap c\,(T)$$

Then since $S \subset A$, $x \in A \cap c\,(T) = c\,(T)$ in A. Therefore, $x \in S \cap T = \emptyset$
a contradiction. Then $S \cap c\,(T) = \emptyset$ and similarly $c\,(S) \cap T = \emptyset$.

$\Longleftarrow$) Suppose that $A = S \cup T$, where $S \cap c\,(T) = c\,(S) \cap T = \emptyset$, and $x \in S$, $y \in T, \forall\, x, y \in X_b, b \in B$. Then

$$\begin{aligned} c\,(S) \text{ in } A &= A \cap c\,(S) \\ &= (S \cup T) \cap c\,(S) \\ &= S \cup \emptyset = \emptyset \end{aligned}$$

Therefore S is closed in A; similary, T is closed in A. Hence, A is disconnected fibrewise.

Proposition 4.3. Suppose A is a connected fibrewise subspace of fibrewise space X with

$$A \subset Y \subset c\,(A)$$

Then Y is also a connected fibrewise subspace of X.

Proof. Let Y be a disconnected fibrewise. Thus, $Y = S \cup T$, where S and T are fibrewise-separated (Proposition 4.2). Since A is connected fibrewise, either $A \subset S$ or $A \subset T$. Suppose $A \subset S$. Then $c\,(A) \subset c\,(S)$. Hence, $Y \subset c\,(A) \subset c\,(S)$. However, $Y = S \cup T$, then $T \subset c\,(S)$. Since $T \cap c\,(S) = \emptyset$, we have arrived at a contradiction. Therefore Y is connected fibrewise.

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