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MODELLING AND STABILITY ANALYSIS OF THE COMPETITIONAL ECOLOGICAL MODEL WITH HARVESTING

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Abstract: The interplay of predation, competition between species and harvesting is one of the most critical aspects of the environment. This paper involves exploring the dynamics of four species' interactions. The system includes two competitive prey and two predators; the first prey is preyed on by the first predator, with the former representing an additional food source for the latter. While the second prey is not exposed to predation but rather is exposed to the harvest. The existence of possible equilibria is found. Conditions of local and global stability for the equilibria are derived. To corroborate our findings, we constructed time series to illustrate the existence and the stability of equilibria numerically by varying the different values of the system's parameters. The results show that system movement could happen around the positive equilibria, if the system stability conditions are met.

Keywords: prey-predator model; competition interaction; local stability; global stability; harvesting.

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1. INTRODUCTION

The main drive of studying population biology and theoretical ecology is to determine the dynamical behaviour mechanisms associated with predator-prey interactions. Much study has been

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undertaken to ascertain the behaviour of species interaction [1]–[3]. The nature of the interaction between the species affects their dynamics. Throughout the two species' interaction, if the growth rate of the first species rises while that of the second decreases, then it will say they are in the prey-predator situation. The first species, called the predator, feeds on other species (prey) [4]. The original fundamental model representing the prey-predator interaction goes back to Lotka and Volterra [5]-[6]. It has now been established as the most straightforward base system for two competitive species. Much ecological research and mathematics have dealt with diverse aspects of prey-predator models [5]–[8].

In the last decades, prey-predator models with more than two species have attracted several scholars [11]–[13]. For example, two prey and one predator have been supposed by Elettreby [14] where the latter can hunt both prey populations. He found the global internal solution for his model. Moreover, in [15], the prey, predator, and super-predator model has been proposed. The predator feeds the prey only in this system, whilst the super predator feeds on the other two species.

The competition between the species and harvesting models has received much attention in various study papers [16]–[21]. For instance, the one prey and two competing predators model has been proposed [22]. It has been found that Hopf bifurcation could be obtained when the consumption rate of the second predator is chosen as a bifurcation parameter.

Harvesting can be used for resource renewal and sustainability from an ecological standpoint. As a result, it is critical to establish sustainable development strategies for harvesting practices; otherwise, overexploitation may abolish some species [17], [23], [24].

In this paper, a four-dimensional model is worked on consisting of two competitive prey, a predator and a super predator. The terms of harvesting the second prey and providing additional food to the predator are also included. The rest of this paper is as follows: In Section 2, the structure of the mathematical model is explained. Section 3 presents the dynamical analysis of the proposed system, including its positivity and boundedness. Then, in Section 4, the local evaluation conditions are calculated for twelve equilibria, while in Section 5, the Lyapunov method is used to prove the global stability of these. Section 6 presents an intensive numerical simulation to delineate some exciting findings associated with the proposed model. Finally, Section 7 reviews the results of our work.

2. Mathematical Model

Suppose an ecological system contains two competitive preys, a predator (first predator) and a top predator (second predator), with the mathematics beings based on the following assumptions. $u_1(t)$ is the density of the first prey at time t and $u_2(t)$ is the density of the second harvested prey at time t, which harms the first prey (and vice versa), whilst $u_3(t)$ and $u_4(t)$ are the densities of the predator and top predator species at time t, respectively.

Under the above assumptions, the following ODEs are obtained:

$$\begin{aligned}
\frac{du_1}{dt} &= r_1 u_1\left(1-\frac{u_1}{k}\right) - \alpha_1 u_1 u_2 - \beta_1 u_1 u_3 = u_1 f_1(u_1,u_2,u_3,u_4),\\
\frac{du_2}{dt} &= r_2 u_2\left(1-\frac{u_2}{l}\right) - \alpha_2 u_1 u_2 - \alpha u_2 = u_2 f_2(u_1,u_2,u_3,u_4),\\
\frac{du_3}{dt} &= r_3 u_3\left(1-\frac{u_3}{m}\right) + \beta_2 u_1 u_3 - \beta_0 u_3 - \gamma_1 u_3 u_4 = u_3 f_3(u_1,u_2,u_3,u_4),\\
\frac{du_4}{dt} &= \gamma_2 u_3 u_4 - \gamma u_4 = u_4 f_4(u_1,u_2,u_3,u_4).
\end{aligned} \tag{1}$$

For all the above parameters $\in (0,\infty)$. Further, Model (1) has been analysed with the initial conditions $u_i(0) \geq 0, i = 1,2,3,4$. $p(u_1) = \beta_1 u_1$ and $p(u_3) = \gamma_1 u_3,$ which are the Lotka-Volterra type of functional responses. The flow chart of system (1) is shown in the following block diagram.

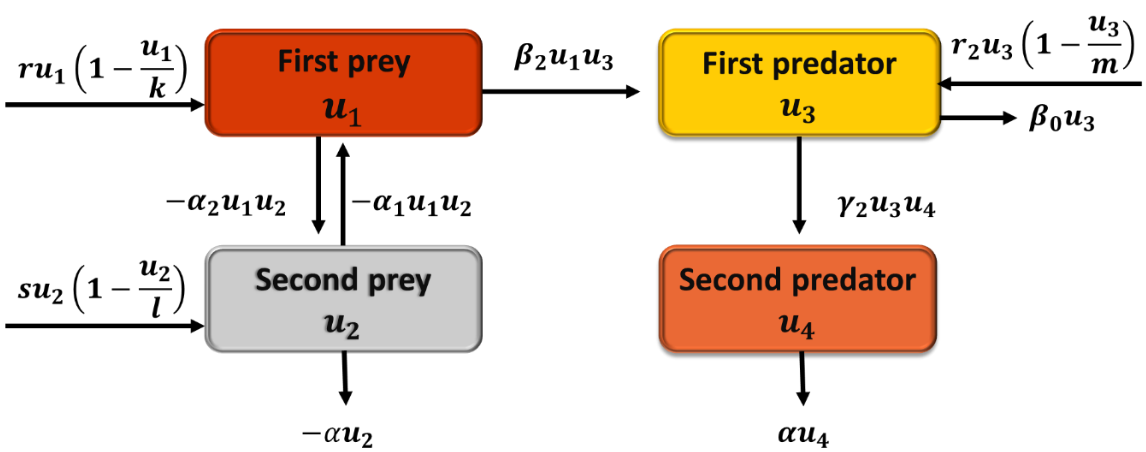

Figure 1: Block diagram for system (1)

In Model (1), we assume that the two prey and first predator reproduce logistically, with intrinsic growth rates r_1, r_2 and r_3 and carrying capacities k, l and m, respectively. $\beta_1$and γ_1 are the attack rate coefficients of the first prey and first predator species due to the first predator and second predator. β_2 and γ_2 are the first prey and first predator biomass conversion rates into the first and second predator, respectively. β_0 and γ represent the first and second predators' natural death rates, respectively. $\alpha_1$is the negative effect on the first prey by the second prey, whilst α_2 is the competition rate of the second prey with the first prey. α represents the harvesting rate of the second prey.

The functions on the right-hand side of the system (1) are $C^1(R_+^4)$ on $R_+^4 = \{(u_1, u_2, u_3, u_4), u_1 \geq 0, u_2 \geq 0, u_3 \geq 0 \geq, u_4 \geq 0\}$ and therefore, they are Lipschitzian. Consequently, the solution of system (1) exists and is unique. Further, all the model (1) solutions with non-negative initial conditions remain positive and bounded, as shown in the following section.

3. POSITIVITY AND BOUNDEDNESS OF THE SOLUTIONS

Theorem 1. All solutions $u_1(t), u_2(t), u_3(t)$ and $u_4(t)$ of the system (1) with the initial conditions $(u_{10}, u_{20}, u_{30}, u_{40}) \in R_+^4$ are positively invariant.

Proof. By integrating the right-hand side of system (1) for u_1, u_2, u_3 and u_4 we get:

$$u_1(t) = u_{10}\, exp\left\{\int_0^t \left[r_1\left(1 - \frac{u_1(s)}{k}\right) - \alpha_1 u_2(s) - \beta_1 u_3(s)\right] ds\right\}$$

$$u_2(t) = u_{20}\, exp\left\{\int_0^t \left[r_2\left(1 - \frac{u_2(s)}{l}\right) - \alpha_2 u_1(s) - \alpha\right] ds\right\}$$

$$u_3(t) = u_{30}\, exp\left\{\int_0^t \left[r_3\left(1 - \frac{u_3(s)}{m}\right) + \beta_2 u_1(s) - \beta_0 - \gamma_1 u_4(s)\right] ds\right\}$$

$$u_4(t) = u_{40}\, exp\left\{\int_0^t [\gamma_2 u_3(s) - \gamma(s)] ds\right\}$$

Then, $u_1(t) > 0$, $u_2(t) > 0$, $u_3(t) > 0$ and $u_4(t) > 0$ for all $t > 0$. Hence, the interior of

R_+^4 is an invariant set of system (1).

Theorem 2. All solutions $u_1(t), u_2(t), u_3(t)$ and $u_4(t)$ of system (1) with the initial conditions (u_1, u_2, u_3, u_4) are uniformly bounded.

Proof: Let $\big(u_1(t), u_2(t), u_3(t), u_4(t)\big)$ be an arbitrary system (1) solution with a non-negative initial condition. Then, for $N(t) = u_1(t) + u_2(t) + u_3(t) + u_4(t)$, we have:

$$\frac{dN}{dt} = \frac{du_1}{dt} + \frac{du_2}{dt} + \frac{du_3}{dt} + \frac{du_4}{dt}$$

$$\frac{dN}{dt} = r_1u_1 - \frac{r_1u_1^2}{k} - \alpha_1u_1u_2 - \beta_1u_1u_3 + r_2u_2 - \frac{r_2u_2^2}{l} - \alpha_2u_1u_2 - \alpha u_2 + r_3u_3 - \frac{r_3u_3^2}{m} + \beta_2u_1u_3 - \beta_0u_3 - \gamma_1u_3u_4 + \gamma_2u_3u_4 - \gamma u_4$$

Hence, $\frac{dN}{dt} + \delta N \leq 2r_1u_1 + 2r_2u_2 + 2r_3u_3 = \tau$, where, $\delta = min.\{r_1, (r_2 + \alpha), (r_3 + \beta_0), \gamma\}$.

Then, by applying Gronwall's Inequality, the following is obtained:

$$0 \leq N\big(u_1(t), u_2(t), u_3(t), u_4(t)\big) \leq \frac{\tau}{\delta}\big(1 - e^{-\delta t}\big) + N(0)e^{-\delta t}$$

Hence, $0 \leq \limsup_{t\to\infty} N(t) \leq \frac{\tau}{\delta}$

Thus, all system (1) solutions that are initiated in R_+^4 are attracted to the region $\sigma = \left\{(u_1, u_2, u_3, u_4) \in R_+^4 : N = u_1 + u_2 + u_3 + u_4 \leq \frac{\tau}{\delta}\right\}$ and thus, the conclusion of the theorem holds.

4. EXISTENCE OF EQUILIBRIA

System (1) has twelve non-negative equilibrium points, namely:

1. The disappearing equilibrium point $F_1 = (0,0,0,0)$.
2. The first predator equilibrium point $F_2 = (0,0,\dot{u}_3,0)$, where $\dot{u}_3 = \frac{m}{r_3}(r_3 - \beta_0)$ exists when

$$r_3 > \beta_0 \qquad (2)$$

3. The second prey equilibrium point $F_3 = (0,\bar{u}_2,0,0)$, where ,$\bar{u}_2 = \frac{l}{r_2}(r_2 - \alpha)$ exists when

$$r_2 > \alpha \tag{3}$$

4. The first prey equilibrium point $F_4 = (k, 0,0,0)$.
5. For the second two species' equilibrium point $F_5 = (0,0, \hat{u}_3, \hat{u}_4)$, where $\hat{u}_3 = \frac{\gamma}{\gamma_2}$ and $\hat{u}_4 = \frac{1}{\gamma_1}\left(r_3 - \frac{r_3\gamma}{m\gamma_2} - \beta_0\right)$. For $\hat{u}_4 > 0$, the following would be the case:

$$r_3 > \frac{r_3\gamma}{m\gamma_2} + \beta_0 \tag{4}$$

6. For the second and third species' equilibrium point $F_6 = (0, \bar{\bar{u}}_2, \bar{\bar{u}}_3, 0)$, where $\bar{\bar{u}}_2 = \frac{l}{r_2}(r_2 - \alpha)$ and $\bar{\bar{u}}_3 = \frac{m}{r_3}(r_3 - \beta_0)$. For $\bar{\bar{u}}_2$ and $\bar{\bar{u}}_3$ to be positive, conditions (2) and (3) must be satisfied.
7. For the first and third species' equilibrium point $F_7 = (\tilde{u}_1, 0, \tilde{u}_3, 0)$, where $\tilde{u}_1 = \frac{1}{\beta_2}\left(\beta_0 + \frac{r_3\tilde{u}_3}{m} - r_3\right)$ and $\tilde{u}_3 = mr_1\left(\frac{k\beta_2 + r_3 - \beta_0}{r_1r_3 + km\beta_1\beta_2}\right)$. It should be clear that for $\tilde{u}_1$ and $\tilde{u}_3$ to be positive, the following must be the case:

$$r_3\left(\frac{m - \tilde{u}_3}{m}\right) < \beta_0 < k\beta_2 + r_3 \tag{5}$$

8. For the first two species' equilibrium point $F_8 = (u_1^\circ, u_2^\circ, 0,0)$, where $u_1^\circ = k\left(\frac{r_1r_2 + \alpha\alpha_1 l - l\alpha_1 r_2}{r_1r_2 - \alpha_1\alpha_2 lk}\right)$ and $u_2^\circ = \frac{1}{\alpha_1}\left(r_1 - \frac{r_1u_1^\circ}{k}\right)$. Clearly, $u_1^\circ > 0$, if one of the following conditions hold:

$$\frac{\alpha_1\alpha_2 lk}{r_1} < r_2 < \frac{r_1r_2 + \alpha\alpha_1 l}{l\alpha_1} \tag{6}$$

$$\frac{r_1r_2 + \alpha\alpha_1 l}{l\alpha_1} < r_2 < \frac{\alpha_1\alpha_2 lk}{r_1} \tag{7}$$

It should also be noted that for u_2° to be positive, the following must be the case:

$$k > u_1^\circ \tag{8}$$

9. For the last three species' equilibrium point $F_9 = (0, u_2', u_3', u_4')$, where $u_2' = \frac{l}{r_2}(r_2 - \alpha)$, $u_3' = \frac{\gamma}{\gamma_2}$ and $u_4' = \frac{1}{\gamma_2}\left(r_3 - \frac{r_3\gamma}{m\gamma_2} - \beta_0\right)$. For u_2' and u_4' to be positive, conditions (3) and (4) must be satisfied.

10. For the second free species' equilibrium point $F_{10} = (u_1'', 0, u_3'', u_4'')$, where: $u_1'' = \frac{k}{r_1}\left(r_1 - \frac{\gamma\beta_1}{\gamma_2}\right)$, $u_3'' = \frac{\gamma}{\gamma_2}$ and $u_4'' = \frac{1}{\gamma_1}\left(r_3 + \beta_2 u_1'' - \frac{r_3 u_3''}{m} - \beta_0\right)$. For u_1'' and u_4'' to be positive, the following must be validated:

$$r_1 > \frac{\gamma\beta_1}{\gamma_2} \tag{9}$$

$$r_3 + \beta_2 u_1'' > \frac{r_3 u_3''}{m} + \beta_0 \tag{10}$$

11. For the first three species' equilibrium point $F_{11} = (\check{u}_1, \check{u}_2, \check{u}_3, 0)$, where $\check{u}_1 = \frac{k(l\alpha_1 r_2 r_3 + m\beta_1 r_2 r_3 - r_1 r_2 r_3 - l\alpha\alpha_1 r_3 - m\beta_0\beta_1 r_2)}{lk\alpha_1\alpha_2 r_3 - r_1 r_2 r_3 - mk\beta_1\beta_2 r_2}$, $\check{u}_2 = \frac{l}{r_2}(r_2 - \alpha_2\check{u}_1 - \alpha)$ and $\check{u}_3 = \frac{m}{r_3}(r_3 + \beta_2\check{u}_1 - \beta_0)$. Clearly, for $\check{u}_1, \check{u}_2$ and $\check{u}_3$ to be positive, condition (13), along with one of the conditions (11) or (12), must be satisfied:

$$\frac{r_1 r_2 r_3 + \alpha_1 r_3 l\alpha + m\beta_0\beta_1 r_2}{l r_3 \alpha_1 + m\beta_1 r_3} < r_2 < \frac{lk\alpha_1\alpha_2 r_3}{r_1 r_3 + \beta_1\beta_2 mk}, \tag{11}$$

$$\frac{lk\alpha_1\alpha_2 r_3}{r_1 r_3 + \beta_1\beta_2 mk} < r_2 < \frac{r_1 r_2 r_3 + \alpha_1 r_3 l\alpha + m\beta_0\beta_1 r_2}{l r_3 \alpha_1 + m\beta_1 r_3}, \tag{12}$$

$$\frac{\beta_0 - r_3}{\beta_2} < \check{u}_1 < \frac{r_2 - \alpha}{\alpha_2}. \tag{13}$$

12. For the positive equilibrium point $F_{12} = (u_1^*, u_2^*, u_3^*, u_4^*)$, where $u_1^* = \frac{1}{\alpha_2}\left(r_2 - \frac{r_2 u_2^*}{l} - \alpha\right)$, $u_2^* = \frac{l}{\gamma_2}\left(\frac{r_1 r_2 \gamma_2 + k\gamma\beta_1\alpha_2 - k r_1\gamma_2\alpha_2 - \alpha r_1\gamma_2}{r_1 r_2 - kl\alpha_1\alpha_2}\right)$, $u_3^* = \frac{\gamma}{\gamma_2}$ and $u_4^* = \frac{1}{\gamma_1}\left(r_3 + \beta_2 u_1^* - \frac{r_3 u_3^*}{m} - \beta_0\right)$. It should be noted that for u_1^*, u_2^* and u_4^* to be positive, conditions (14) and (15), along with one of the conditions (16) or (17), must be satisfied:

$$r_2 > \frac{r_2 u_2^*}{l} + \alpha, \tag{14}$$

$$r_3 + \beta_2 u_1^* > \frac{r_3 u_3^*}{m} + \beta_0, \tag{15}$$

$$\frac{kl\alpha_1\alpha_2}{r_2} < r_1 < \frac{r_1 r_2 \gamma_2 + k\gamma\beta_1\alpha_2}{k\gamma_2\alpha_2 + \gamma_2\alpha}, \tag{16}$$

$$\frac{r_1 r_2 \gamma_2 + k\gamma\beta_1\alpha_2}{k\gamma_2\alpha_2 + \gamma_2\alpha} < r_1 < \frac{kl\alpha_1\alpha_2}{r_2}. \tag{17}$$

5. LOCAL STABILITY

This section explores the local stability behaviour of system (1) 's equilibrium points.

The Jacobin matrix of system (1) at any point, say (u_1, u_2, u_3, u_4), can be written as:

$$J = \begin{bmatrix} u_1 \frac{\partial f_1}{\partial u_1} + f_1 & u_1 \frac{\partial f_1}{\partial u_2} & u_1 \frac{\partial f_1}{\partial u_3} & u_1 \frac{\partial f_1}{\partial u_4} \\ u_2 \frac{\partial f_2}{\partial u_1} & u_2 \frac{\partial f_2}{\partial u_2} + f_2 & u_2 \frac{\partial f_2}{\partial u_3} & u_2 \frac{\partial f_2}{\partial u_4} \\ u_3 \frac{\partial f_3}{\partial u_1} & u_3 \frac{\partial f_3}{\partial u_2} & u_3 \frac{\partial f_3}{\partial u_3} + f_3 & u_3 \frac{\partial f_3}{\partial u_4} \\ u_4 \frac{\partial f_4}{\partial u_1} & u_4 \frac{\partial f_4}{\partial u_2} & u_4 \frac{\partial f_4}{\partial u_3} & u_4 \frac{\partial f_4}{\partial u_4} + f_4 \end{bmatrix} = (a_{ij})_{4\times 4},$$

where, $a_{11} = r_1 - \frac{2r_1u_1}{k} - \alpha_1 u_2 - \beta_1 u_3$, $a_{12} = -\alpha_1 u_1$, $a_{13} = -\beta_1 u_1$, $a_{14} = 0$, $a_{21} = -\alpha_2 u_2$, $a_{22} = r_2 - \frac{2r_2u_2}{l} - \alpha_2 u_1 - \alpha$, $a_{23} = a_{24} = 0$, $a_{31} = \beta_2 u_3$, $a_{32} = 0$, $a_{33} = r_3 - \frac{2r_3u_3}{m} + \beta_2 u_1 - \beta_0 - \gamma_1 u_4$, $a_{34} = -\gamma_1 u_3$, $a_{41} = a_{42} = 0$, $a_{43} = \gamma_2 u_4$ and $a_{44} = \gamma_2 u_3 - \gamma$.

Consequently, the following is obtained.

1. The Jacobian matrix at $F_1 = (0,0,0,0)$ is given as:

$$J(F_1) = \begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 - \alpha & 0 & 0 \\ 0 & 0 & r_3 - \beta_0 & 0 \\ 0 & 0 & 0 & -\gamma \end{bmatrix}. \tag{18}$$

Then, $J(F_1)$ has the eigenvalues $\lambda_{11} = r_1 > 0$, $\lambda_{12} = r_2 - \alpha$, $\lambda_{13} = r_3 - \beta_0$ and $\lambda_{14} = -\gamma < 0$, which means F_1 is a saddle point.

2. The Jacobian matrix at $F_2 = (0,0,\dot{u}_3,0)$ can be written as:

$$J(F_2)=\begin{bmatrix} r_1-\beta_1\dot{u}_3 & 0 & 0 & 0 \\ 0 & r_2-\alpha & 0 & 0 \\ \beta_2\dot{u}_3 & 0 & -r_3+\beta_0 & -\gamma_1\dot{u}_3 \\ 0 & 0 & 0 & \gamma_2\dot{u}_3-\gamma \end{bmatrix}. \quad (19)$$

Then, $J(F_2)$ has the eigenvalues $\lambda_{21}=r_1-\beta_1\dot{u}_3, \lambda_{22}=r_2-\alpha, \lambda_{23}=-r_3+\beta_0<0$ and $\lambda_{24}=\gamma_2\dot{u}_3-\gamma$. Clearly, λ_{23} is negative whenever F_2 exists. That means F_2 is a locally asymptotical stable point, if and only if, the following conditions are satisfied:

$$r_2<\alpha, \quad (20)$$

$$\frac{r_1}{\beta_1}<\dot{u}_3<\frac{\gamma}{\gamma_2} \quad (21)$$

3. The Jacobian matrix at $F_3=(0,\bar{u}_2,0,0)$ can be written as:

$$J(F_3)=\begin{bmatrix} r_1-\alpha_1\bar{u}_2 & 0 & 0 & 0 \\ -\alpha_2\bar{u}_2 & -r_2+\alpha & 0 & 0 \\ 0 & 0 & r_3-\beta_0 & 0 \\ 0 & 0 & 0 & -\gamma \end{bmatrix}. \quad (22)$$

Then, $J(F_3)$ has the eigenvalues $\lambda_{31}=r_1-\alpha_1\bar{u}_2$, $\lambda_{32}=-r_2+\alpha<0$, $\lambda_{33}=r_3-\beta_0$ and $\lambda_{34}=-\gamma<0$. Clearly, λ_{32} is negative whenever F_3 exists. That means F_3 is a locally asymptotical stable point, if and only if the following conditions hold:

$$r_1<\alpha_1\bar{u}_2, \quad (23)$$

$$r_3<\beta_0. \quad (24)$$

4. The Jacobian matrix at $F_4=(k,0,0,0)$ can be written as:

$$J(F_4)=\begin{bmatrix} -r_1 & -\alpha k & -\beta_1 k & 0 \\ 0 & r_2-\alpha_2 k-\alpha & 0 & 0 \\ 0 & 0 & r_3+\beta_2 k-\beta_0 & 0 \\ 0 & 0 & 0 & -\gamma \end{bmatrix}. \quad (25)$$

Then, $J(F_4)$ has the eigenvalues $\lambda_{41}=-r_1<0$, $\lambda_{42}=r_2-\alpha_2 k-\alpha$, $\lambda_{43}=r_3+\beta_2 k-\beta_0$ and $\lambda_{44}=-\gamma<0$. That means F_4 is a locally asymptotical stable point provided that:

$$\frac{r_2-\alpha}{\alpha_2}<k<\frac{\beta_0-r_3}{\beta_2}. \quad (26)$$

5. The Jacobian matrix at $F_5=(0,0,\hat{u}_3,\hat{u}_4)$ can be written as:

$$J(F_5)=\begin{bmatrix} r_1-\beta_1\hat{u}_3 & 0 & 0 & 0 \\ 0 & r_2-\alpha & 0 & 0 \\ \beta_2\hat{u}_3 & 0 & \frac{-r_3\gamma}{m\gamma_2} & -\frac{\gamma_1\gamma}{\gamma_2} \\ 0 & 0 & \gamma_2\hat{u}_4 & 0 \end{bmatrix}. \qquad (27)$$

Then, the characteristic equation of $J(F_5)$ is given by:

$$(r_1-\beta_1\hat{u}_3-\lambda)(r_2-\alpha-\lambda)\left(\lambda^2+\left(\frac{r_3\gamma}{m\gamma_2}\right)\lambda+\gamma_1\gamma\hat{u}_4\right). \qquad (28)$$

The eigenvalues of Eq. (28) can be written as follows: $\lambda_{51}=r_1-\beta_1\hat{u}_3$, $\lambda_{52}=r_2-\alpha$, $\lambda_{53}+\lambda_{54}=\frac{-r_3\gamma}{m\gamma_2}<0$ and $\lambda_{53}.\lambda_{54}=\gamma_1\gamma\hat{u}_4>0$. That means F_5 is a locally asymptotical stable point provided that condition (20) is satisfied along with the following:

$$r_1<\beta_1\hat{u}_3. \qquad (29)$$

6. The Jacobian matrix at $F_6=(0,\bar{\bar{u}}_2,\bar{\bar{u}}_3,0)$ can be written as:

$$J(F_6)=\begin{bmatrix} r_1-\alpha_1\bar{\bar{u}}_2-\beta_1\bar{\bar{u}}_3 & 0 & 0 & 0 \\ -\alpha_2\bar{\bar{u}}_2 & -r_2+\alpha & 0 & 0 \\ \beta_2\bar{\bar{u}}_3 & 0 & -r_3+\beta_0 & -\gamma_1\bar{\bar{u}}_3 \\ 0 & 0 & 0 & \gamma_2\bar{\bar{u}}_3-\gamma \end{bmatrix}. \qquad (30)$$

Then the eigenvalues of $J(F_6)$ are given by $\lambda_{61}=r_1-\alpha_1\bar{\bar{u}}_2-\beta_1\bar{\bar{u}}_3$, $\lambda_{62}=-r_2+\alpha<0, \lambda_{63}=-r_3+\beta_0<0$ and $\lambda_{64}=\gamma_2\bar{\bar{u}}_3-\gamma$. That means F_6 is a locally asymptotical stable point provided that the existing conditions (2) and (3) are satisfied along with:

$$\frac{r_1-\alpha_1\bar{\bar{u}}_2}{\beta_1}<\bar{\bar{u}}_3<\frac{\gamma}{\gamma_2}. \qquad (31)$$

7. The Jacobian matrix at $F_7=(\tilde{u}_1,0,\tilde{u}_3,0)$ can be written as:

$$J(F_7)=\begin{bmatrix} \frac{-r_1\tilde{u}_1}{k} & -\alpha_1\tilde{u}_1 & -\alpha_1\tilde{u}_1 & 0 \\ 0 & r_2-\alpha_2\tilde{u}_1-\alpha & 0 & 0 \\ \beta_2\tilde{u}_3 & 0 & \frac{-r_3\tilde{u}_3}{m} & -\gamma_1\tilde{u}_3 \\ 0 & 0 & 0 & \gamma_2\tilde{u}_3-\gamma \end{bmatrix}. \qquad (32)$$

Then, the eigenvalues of $J(F_7)$ are given by $\lambda_{71}+\lambda_{73}=-\left(\frac{r_1\tilde{u}_1}{k}+\frac{r_3\tilde{u}_3}{m}\right)<0$, $\lambda_{71}.\lambda_{73}=\frac{r_1r_3\tilde{u}_1\tilde{u}_3}{km}+\beta_1\beta_2\tilde{u}_1\tilde{u}_3>0$, $\lambda_{72}=r_2-\alpha_2\tilde{u}_1-\alpha$ and $\lambda_{74}=\gamma_2\tilde{u}_3-\gamma$. That means F_7 is a locally asymptotical stable point provided that:

$$r_2<\alpha_2\tilde{u}_1+\alpha \tag{33}$$

$$\gamma_2\tilde{u}_3<\gamma \tag{34}$$

8. The Jacobian matrix at $F_8=(u_1^\circ,u_2^\circ,0,0)$ can be written as:

$$J(F_8)=\begin{bmatrix} \frac{-r_1u_1^\circ}{k} & -\alpha_1u_1^\circ & -\beta_1u_1^\circ & 0 \\ -\alpha_2u_2^\circ & \frac{-r_2u_2^\circ}{l} & 0 & 0 \\ 0 & 0 & r_3+\beta_2u_1^\circ-\beta_0 & 0 \\ 0 & 0 & 0 & -\gamma \end{bmatrix}. \tag{35}$$

Then, the eigenvalues of $J(F_8)$ are given by $\lambda_{81}+\lambda_{82}=-\left(\frac{r_1u_1^\circ}{k}+\frac{r_2u_2^\circ}{l}\right)<0$, $\lambda_{81}.\lambda_{82}=u_1^\circ u_2^\circ\left[\left(\frac{r_1r_2}{kl}\right)-\alpha_1\alpha_2\right]>0$, $\lambda_{83}=r_3+\beta_2u_1^\circ-\beta_0$ and $\lambda_{84}=-\gamma<0$. Clearly, $\lambda_{81}.\lambda_{82}$ is positive whenever F_8 exists. That means F_8 is a locally asymptotical stable point provided that:

$$r_3+\beta_2u_1^\circ<\beta_0. \tag{36}$$

9. The Jacobian matrix at $F_9=(0,u_2',u_3',u_4')$ can be written as:

$$J(F_9)=\begin{bmatrix} r_1-\alpha_1u_2'-\beta_1u_3' & 0 & 0 & 0 \\ -\alpha_2u_2' & -r_2+\alpha & 0 & 0 \\ \beta_2u_3' & 0 & \frac{-r_3\gamma}{m\gamma_2} & \frac{-\gamma\gamma_1}{\gamma_2} \\ 0 & 0 & \gamma_2u_4' & 0 \end{bmatrix}. \tag{37}$$

Then, the eigenvalues of $J(F_9)$ are given by $\lambda_{91}=r_1-\alpha_1u_2'-\beta_1u_3'$, $\lambda_{92}=-r_2+\alpha<0$, $\lambda_{93}+\lambda_{94}=\frac{-r_3\gamma}{m\gamma_2}<0$ and $\lambda_{93}.\lambda_{94}=\gamma\gamma_1u_4'>0$. That means F_9 is a locally asymptotical stable point provided that the existing condition (3) is satisfied along with the following:

$$r_1<\alpha_1u_2'+\beta_1u_3' \tag{38}$$

10. The Jacobian matrix at $F_{10}=(u_1'',0,u_3'',u_4'')$ can be written as:

$$J(F_{10}) = \begin{bmatrix} \frac{-r_1 u_1''}{k} & -\alpha_1 u_1'' & -\beta_1 u_1'' & 0 \\ 0 & r_2 - \alpha_2 u_1'' - \alpha & 0 & 0 \\ \beta_2 u_3'' & 0 & \frac{-r_3 u_3''}{m} & \frac{-\gamma\gamma_1}{\gamma_2} \\ 0 & 0 & \gamma_2 u_4'' & 0 \end{bmatrix}. \quad (39)$$

The first root of the characteristic equation of $J(F_{10})$ is $r_2 - \alpha_2 u_1'' - \alpha$ and the other three roots are given by:

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0 \quad (40)$$

where, $A_1 = \frac{r_1 u_1''}{k} + \frac{r_3 u_3''}{m} > 0$

$$A_2 = \frac{r_1 r_3 u_1'' u_3''}{mk} + \beta_1\beta_2 u_1'' u_3'' + \gamma\gamma_1 u_4''$$

$$A_3 = \frac{r_1\gamma\gamma_1 u_1'' u_4''}{k} > 0$$

$$\Delta = A_1A_2 - A_3 = \left[\frac{r_1 u_1''}{k} + \frac{r_3 u_3''}{m}\right]\left[\beta_1\beta_2 u_1'' u_3'' + \frac{r_1 r_3 u_1'' u_3''}{mk}\right] + \frac{r_3\gamma\gamma_1 u_3'' u_4''}{m} > 0$$

Now, according to the Routh-Hurwitz criteria [10], all the eigenvalues of $J(F_{10})$ have roots with negative real parts, on condition that $A_1 > 0, A_3 > 0$ and $\Delta > 0$. Then, the stability of F_{10} depends on the sign of the first root. So, it can be concluded that F_{10} is a locally asymptotical stable point if:

$$r_2 < \alpha_2 u_1'' + \alpha \quad (41)$$

11. The Jacobian matrix at $F_{11} = (\check{u}_1, \check{u}_2, \check{u}_3, 0)$ can be written as:

$$J(F_{11}) = \begin{bmatrix} \frac{-r_1\check{u}_1}{k} & -\alpha_1\check{u}_1 & -\beta_1\check{u}_1 & 0 \\ -\alpha_2\check{u}_2 & \frac{-r_2\check{u}_2}{l} & 0 & 0 \\ \beta_2\check{u}_3 & 0 & \frac{-r_3\check{u}_3}{m} & -\gamma_1\check{u}_3 \\ 0 & 0 & 0 & \gamma_2\check{u}_3 - \gamma \end{bmatrix}. \quad (42)$$

The first root of the characteristic equation of $J(F_{11})$ is $\gamma_2\check{u}_3 - \gamma$ and the other three are given by:

$$\lambda^3 + A_{11}\lambda^2 + A_{22}\lambda + A_{33} = 0 \tag{43}$$

Here, $A_{11} = \frac{r_1 \check{u}_1}{k} + \frac{r_2 \check{u}_2}{l} + \frac{r_3 \check{u}_3}{m} > 0, A_{22} = \frac{r_1 r_3 \check{u}_1 \check{u}_3}{mk} + \frac{r_2 r_3 \check{u}_2 \check{u}_3}{ml} + \frac{r_1 r_2 \check{u}_1 \check{u}_2}{kl} + \beta_1\beta_2 \check{u}_1 \check{u}_3 - \alpha_1\alpha_2 \check{u}_1 \check{u}_2$, $A_{33} = \frac{r_1 r_2 r_3 \check{u}_1 \check{u}_2 \check{u}_3}{mkl} + \frac{\beta_1\beta_2 \check{u}_1 \check{u}_2 \check{u}_3 r_2}{l} - \frac{\alpha_1\alpha_2 \check{u}_1 \check{u}_2 \check{u}_3 r_3}{m}$ and $\breve{\Delta} = A_{11}A_{22} - A_{33} = \left(\frac{r_1 \check{u}_1}{k} + \frac{r_3 \check{u}_3}{m}\right)\left(\frac{r_2^2 \check{u}_2^2}{l^2} + \beta_1\beta_2 \check{u}_1 \check{u}_3\right) + \frac{r_1^2 \check{u}_1^2}{k^2}\left(\frac{r_3 \check{u}_3}{m} + \frac{r_2 \check{u}_2}{l}\right) + \frac{2 r_1 r_2 r_3 \check{u}_1 \check{u}_2 \check{u}_3}{mkl} + \left(\frac{r_1 \check{u}_1}{k} + \frac{r_2 \check{u}_2}{l}\right)\left(\frac{r_3^2 \check{u}_3^2}{m^2} - \alpha_1\alpha_2 \check{u}_1 \check{u}_2\right)$

According to the Routh-Hurwitz criteria, all the eigenvalues of (43) have roots with negative real parts; if $A_{11} > 0, A_{33} > 0$ and $\breve{\Delta} > 0$. Therefore, F_{11} is a locally asymptotical stable point, if:

$$\frac{r_1 r_2 r_3}{mkl} + \frac{\beta_1\beta_2 r_2}{l} > \frac{\alpha_1\alpha_2 r_3}{m} \tag{44}$$

$$r_3^2 \check{u}_3^2 \geq \alpha_1\alpha_2 \check{u}_1 \check{u}_2 m^2 \tag{45}$$

$$\gamma_2 \check{u}_3 < \gamma \tag{46}$$

12. The Jacobian matrix at $F_{12} = (u_1^*, u_2^*, u_3^*, u_4^*)$ can be written as:

$$J(F_{12}) = \left(b_{ij}\right)_{4\times 4} \tag{47}$$

where, $b_{11} = \frac{-r_1 u_1^*}{k}$, $b_{12} = -\alpha_1 u_1^*$, $b_{13} = -\beta_1 u_1^*$, $b_{14} = 0$, $b_{21} = -\alpha_2 u_2^*$, $b_{22} = \frac{-r_2 u_2^*}{l}$, $b_{23} = b_{24} = 0$, $b_{31} = \beta_2 u_3^*$, $b_{32} = 0$, $b_{33} = \frac{-r_3 u_3^*}{m}$, $b_{34} = -\gamma_1 u_3^*$, $b_{41} = b_{42} = b_{44} = 0$ and $b_{43} = \gamma_2 u_4^*$.

So, the characteristic equation of $J(F_{12})$ can be written as:

$$\lambda^4 + B_1\lambda^3 + B_2\lambda^2 + B_3\lambda + B_4 = 0, \tag{48}$$

where:

$B_1 = -(M_1 + b_{33}) > 0$, $B_2 = -M_2 + b_{33}M_1 - M_3$,

$B_3 = b_{33}M_2 + b_{34}b_{43}M_1 + b_{22}b_{13}b_{31} > 0, B_4 = b_{34}b_{43}M_2 > 0$,

$\Delta_1 = B_1B_2 - B_3 = M_1M_2 - b_{33}M_1^2 - b_{33}^2 M_1 + b_{33}M_3 + b_{11}b_{13}b_{31} > 0$, and

$$\Delta_2 = B_3\Delta_1 - B_1^2B_4$$

$$= [M_2 - b_{33}M_1](b_{33}M_1M_2 + b_{22}b_{13}b_{31}M_1) - M_1^2b_{34}b_{43}b_{33}(M_1 + b_{33})$$
$$+ [b_{33}M_2 + b_{22}b_{13}b_{31}](b_{11}b_{13}b_{31} - b_{33}^2M_1) + b_{34}b_{43}M_1(b_{33}M_3 + b_{11}b_{13}b_{31})$$
$$+ b_{33}^2M_2(M_3 - b_{34}b_{43}) + b_{33}(b_{22}b_{13}b_{31}M_3 - 2b_{34}b_{43}M_1M_2).$$

Now, according to the Routh-Hurwitz criteria, all the eigenvalues of $J(F_{12})$ have roots with negative real parts, provided that $B_1 > 0, B_3 > 0, B_4 > 0, \Delta_1 > 0$ and $\Delta_2 > 0$. Therefore, F_{12} is a locally asymptotical stable point, if $b_{11}b_{13}b_{31} \geq 2b_{33}M_2$,

$$M_3 > max.\left\{b_{34}b_{43}, \frac{2b_{34}b_{43}M_1M_2}{b_{22}b_{13}b_{31}}\right\}, \tag{49}$$

where, $M_1 = b_{11} + b_{22} = -\left(\frac{r_1u_1^*}{k} + \frac{r_2u_2^*}{l}\right) < 0$, $M_2 = b_{12}b_{21} - b_{11}b_{22} = u_1^*u_2^*\left(\alpha_1\alpha_2 - \frac{r_1r_2}{kl}\right)$ and $M_3 = b_{13}b_{31} + b_{34}b_{43} = -(\beta_1\beta_2u_1^*u_3^* + \gamma_1\gamma_2u_3^*u_4^*) < 0.$ Clearly, M_2 is negative provided that the existing condition (16) is satisfied. Condition (49) indicates the carrying capacity threshold values that the predator species may service.

6. GLOBAL DYNAMICAL BEHAVIOUR

This section discusses the conditions of the global stability property of the system's (1) equilibria using the Lyapunov method.

Theorem 3 Assume the local stability conditions (20) and (21) hold, then the first predator equilibrium point $F_2 = (0,0,\dot{u}_3,0)$ of system (1) is globally stable.

Proof: Define $W_2 = c_1u_1 + c_2u_2 + c_3\left(u_3 - \dot{u}_3 - \dot{u}_3\ln\frac{u_3}{\dot{u}_3}\right) + c_4u_4$, where c_1, c_2 , c_3 and c_4 are positive constants to be determined. $W_2(u_1, u_2, u_3, u_4)$ is a positive definite about F_2.

Thus,

$$\frac{dw_2}{dt} = c_1u_1\left(r_1\left(1 - \frac{u_1}{k}\right) - \alpha_1u_2 - \beta_1u_3\right) + c_2u_2\left(r_2\left(1 - \frac{u_2}{l}\right) - \alpha_2u_1 - \alpha\right) + c_3(u_3 -$$
$$\dot{u}_3)\left(r_3\left(1 - \frac{u_3}{m}\right) + \beta_2u_1 - \beta_0 - \gamma_1u_4\right) + c_4u_4(\gamma_2u_3 - \gamma)$$

i.e.,

$$\frac{dw_2}{dt} = -\frac{c_3 r_3}{m}(u_3 - \dot{u}_3)^2 - \left(\frac{c_1 r_1 u_1^2}{k}\right) - \left(\frac{c_2 r_2 u_2^2}{l}\right) + (c_4\gamma_2 - c_3\gamma_1)u_3u_4 + (c_3\beta_2 - c_1\beta_1)u_1u_3$$
$$+ c_1u_1r_1 - c_1\alpha_1u_1u_2 + c_2u_2r_2 - c_2\alpha_2u_1u_2$$

By choosing the positive constants as: $c_1 = \frac{\beta_2}{\beta_1}, c_4 = \frac{\gamma_1}{\gamma_2}, c_2 = c_3 = 1,$ the following is obtained,

$$\frac{dw_2}{dt} = -\frac{r_3(u_3-\dot{u}_3)^2}{m} - \left(\frac{\beta_2 r_1 u_1^2}{\beta_1 k}\right) - \left(\frac{r_2 u_2^2}{l}\right) + u_1\beta_2\left(\frac{r_1}{\beta_1} - \dot{u}_3\right) + u_2(r_2 - \alpha) + u_4\gamma_1\left(\dot{u}_3 - \frac{\gamma}{\gamma_2}\right).$$

Then, $\frac{dw_2}{dt} < 0$ under the local stability conditions (20) and (21). Therefore, F_2 is GAS in R_+^4.

Theorem 4 Assume the second prey equilibrium point $F_3 = (0, \bar{u}_2, 0,0)$ exists. Then, the basin of attraction of F_3 is the sub-region of R_+^4 which can be defined as:

$$\vartheta = \left\{(u_1, u_2, u_3, u_4): u_1 \geq 0, u_2 \geq max.\left\{\frac{\gamma_1}{\alpha_1}, \bar{u}_2\right\} \geq 0, u_4 \geq 0\right\}$$

Proof: Define $w_3 = c_1u_1 + c_2\left(u_2 - \bar{u}_2 - \bar{u}_2 \ln\frac{u_2}{\bar{u}_2}\right) + c_3u_3 + c_4u_4$, where c_1, c_2, c_3 and c_4 are positive constants to be determined. $W_3(u_1, u_2, u_3, u_4)$ is a positive definite about F_3 and Thus:

$$\frac{dw_3}{dt} = c_1u_1\left(r_1\left(1 - \frac{u_1}{k}\right) - \alpha_1u_2 - \beta_1u_3\right)$$
$$+ c_2(u_2 - \bar{u}_2)\left(\frac{-r_2}{l}(u_2 - \bar{u}_2) - \alpha_2u_1\right)$$
$$+ c_3u_3\left(r_3\left(1 - \frac{u_3}{m}\right) + \beta_2u_1 - \beta_0 - \gamma_1u_4\right) + c_4u_4(\gamma_2u_3 - \gamma)$$

i.e.,

$$\frac{dw_3}{dt} = -\frac{c_1r_1u_1^2}{k} - \frac{c_2r_2(u_2 - \bar{u}_2)^2}{l} - \frac{c_3r_1u_3^2}{m} + (c_4\gamma_2 - c_3\gamma_1)u_3u_4 + (c_3\beta_2 - c_1\beta_1)u_1u_3$$
$$+ c_1u_1(\alpha_2u_2 - r_1) - c_2\alpha_2u_1(u_2 - \bar{u}_2) - c_3u_3r_3 - c_3\beta_0u_3 - c_4\gamma u_4$$

By choosing the positive constants as: $c_1 = \frac{\beta_2}{\beta_1}, c_4 = \frac{\gamma_1}{\gamma_2}, c_2 = c_3 = 1,$ the following is obtained,

$$\frac{dw_3}{dt} = -\frac{\beta_2 r_1 u_1^2}{\beta_1 k} - \frac{r_2(u_2-\bar{u}_2)^2}{l} - \frac{r_1u_3^2}{m} - \frac{\beta_2u_1(\alpha_2u_2 - r_1)}{\beta_1} - \alpha_2u_1(u_2 - \bar{u}_2) - u_3r_3 - \beta_0u_3 - \frac{\gamma\gamma_1u_4}{\gamma_2}.$$

Then, $\frac{dw_3}{dt} < 0$ and hence, w_3 is a Lyapunov function. Therefore, any solution starting in ϑ approaches asymptotically to F_3.

Theorem 5 Assume the local stability condition (26) along with the following:

$$\alpha_1 k + r_2 < \alpha \tag{50}$$

is satisfied, then $F_4 = (k, 0,0,0)$ is globally stable.

Proof: Define $W_4 = c_1\left(u_1 - k - k\ln\frac{u_1}{k}\right) + c_2u_2 + c_3u_3 + c_4u_4$, then:

$$\frac{dw_4}{dt} = c_1(u_1 - k)\left(\frac{-r_1(u_1-k)}{k} - \alpha_1u_2 - \beta_1u_3\right) + c_2u_2\left(r_2\left(1-\frac{u_2}{l}\right) - \alpha_2u_1 - \alpha\right)$$
$$+ c_3u_3\left(r_3\left(1-\frac{u_3}{m}\right) + \beta_2u_1 - \beta_0 - \gamma_1u_4\right) + c_4u_4(\gamma_2u_3 - \gamma)$$

i.e.,

$$\frac{dw_4}{dt} = -\frac{c_1r_1(u_1-k)^2}{k} - c_1\alpha_1u_1u_2 + c_1\alpha_1ku_2 + c_1\beta_1ku_3 - (\gamma_1c_3 - \gamma_2c_4)u_3u_4$$
$$-(c_1\beta_1 - c_3\beta_2)u_1u_3 + c_2r_2u_2 - \frac{c_2r_2u_2^2}{l} - c_2\alpha_2u_1u_2 - c_2\alpha u_2 + c_3u_3r_3$$
$$-\frac{c_3r_3u_3^2}{m} - c_3\beta_0u_3 - c_4\gamma u_4$$

By choosing the positive constants as: $c_1 = c_2 = 1, c_3 = \frac{\beta_1}{\beta_2}, c_4 = \frac{\beta_1\gamma_1}{\beta_2\gamma_2}$, the following is obtained,

$$\frac{dw_4}{dt} = -\frac{r_1(u_1-k)^2}{k} - \frac{r_2u_2^2}{l} - \frac{\beta_1r_3u_3^2}{\beta_2m} + \beta_1u_3\left(\frac{\beta_2k + r_3 - \beta_0}{\beta_2}\right) + u_2(\alpha_1k + r_2 - \alpha)$$
$$- u_1u_2(\alpha_1 + \alpha_2) - \frac{\gamma\gamma_1\beta_1u_4}{\gamma_2\beta_2}$$

Then, $\frac{dw_4}{dt} < 0$ under conditions (26) and (50). Therefore, F_4 is globally stable in R_+^4.

Theorem 6 Assume the local stability conditions (20) and (29) hold, then $F_5 = (0,0,\hat{u}_3,\hat{u}_4)$ is globally stable.

Proof: Define $W_5 = c_1u_1 + c_2u_2 + c_3\left(u_3 - \hat{u}_3 - \hat{u}_3\ln\frac{u_3}{\hat{u}_3}\right) + c_4\left(u_4 - \hat{u}_4 - \hat{u}_4\ln\frac{u_4}{\hat{u}_4}\right)$, then

$$\frac{dw_5}{dt} = c_1u_1\left(r_1\left(1-\frac{u_1}{k}\right) - \alpha_1u_2 - \beta_1u_3\right) + c_2u_2\left(r_2\left(1-\frac{u_2}{l}\right) - \alpha_2u_1 - \alpha\right)$$
$$+ c_3(u_3 - \hat{u}_3)\left(-\frac{r_3(u_3-\hat{u}_3)}{m} + \beta_2u_1 - \gamma_1(u_4 - \hat{u}_4)\right)$$
$$+ c_4(u_4 - \hat{u}_4)\left(\gamma_2(u_3 - \hat{u}_3)\right)$$

i.e.,

$$\frac{dw_5}{dt} = c_1u_1r_1 - \frac{c_1r_1u_1^2}{k} - c_1\alpha_1u_1u_2 - (c_1\beta_1 + c_3\beta_2)u_1u_3 + c_2r_2u_2 - \frac{c_2r_2u_2^2}{l} - c_2\alpha_2u_1u_2 - c_2\alpha u_2 - \frac{c_3r_3(u_3-\hat{u}_3)^2}{m} - c_3\beta_2u_1\hat{u}_3 - (c_3\gamma_1 + c_4\gamma_2)(u_3-\hat{u}_3)(u_4-\hat{u}_4)$$

By choosing the positive constants as: $c_2 = c_3 = 1, c_1 = \frac{\beta_2}{\beta_1}, c_4 = \frac{\gamma_1}{\gamma_2}$, the following is obtained:

$$\frac{dw_5}{dt} = -\frac{\beta_2r_1u_1^2}{\beta_1k} - \frac{r_2u_2^2}{l} - \frac{r_3(u_3-\hat{u}_3)^2}{m} + \left(\frac{r_1}{\beta_1} - \hat{u}_3\right)\beta_2u_1 + (r_2-\alpha)u_2 - \left(\frac{\alpha_1\beta_2}{\beta_1} + \alpha_2\right)u_1u_2$$

Then, $\frac{dw_5}{dt} < 0$ and hence, w_5 is a Lyapunov function under the local stability conditions (20) and (29). Therefore, F_5 is globally stable in R_+^4.

Theorem 7 Assume the local stability condition (31) and the following:

$$r_1 + \alpha_2\bar{\bar{u}}_2 < \beta_2\bar{\bar{u}}_3, \tag{51}$$

is satisfied, then $F_6 = (0, \bar{\bar{u}}_2, \bar{\bar{u}}_3, 0)$ is globally stable.

Proof: Define $W_6 = c_1u_1 + c_2\left(u_2 - \bar{\bar{u}}_2 - \bar{\bar{u}}_2\ln\frac{u_2}{\bar{\bar{u}}_2}\right) + c_3\left(u_3 - \bar{\bar{u}}_3 - \bar{\bar{u}}_3\ln\frac{u_3}{\bar{\bar{u}}_3}\right) + c_4u_4$, then

$$\frac{dw_6}{dt} = c_1u_1\left(r_1\left(1-\frac{u_1}{k}\right) - \alpha_1u_2 - \beta_1u_3\right) + c_2(u_2-\bar{\bar{u}}_2)\left(-\frac{r_2(u_2-\bar{\bar{u}}_2)}{l} - \alpha_2u_1\right) + c_3(u_3-\bar{\bar{u}}_3)\left(-\frac{r_3(u_3-\bar{\bar{u}}_3)}{m} + \beta_2u_1 - \gamma_1u_4\right) + c_4u_4(\gamma_2u_3-\gamma)$$

i.e.,

$$\frac{dw_6}{dt} = c_1u_1r_1 - \frac{c_1r_1u_1^2}{k} - c_1\alpha_1u_1u_2 - \frac{r_2}{l}c_2(u_2-\bar{\bar{u}}_2)^2 + (c_3\beta_2 - c_1\beta_1)u_1u_3 - c_2\alpha_2u_1u_2 + c_2\alpha_2u_1\bar{\bar{u}}_2 - \frac{c_3r_3(u_3-\bar{\bar{u}}_3)^2}{m} - c_3\beta_2u_1\bar{\bar{u}}_3 + (c_4\gamma_2 - c_3\gamma_1)u_3u_4 + c_3\gamma_1\bar{\bar{u}}_3u_4 - c_4u_4\gamma$$

By choosing the positive constants as: $c_1 = c_2 = 1, c_3 = \frac{\beta_1}{\beta_2}, c_4 = \frac{\gamma_1\beta_1}{\gamma_2\beta_2}$, the following is obtained:

$$\frac{dw_6}{dt} = -\frac{r_1u_1^2}{k} + -\frac{r_2(u_2-\bar{\bar{u}}_2)^2}{l} - \frac{r_3\beta_1(u_3-\bar{\bar{u}}_3)^2}{m\beta_2} + u_1(r_1 + \alpha_2\bar{\bar{u}}_2 - \beta_2\bar{\bar{u}}_3) - (\alpha_1+\alpha_2)u_1u_2 + \frac{\beta_1\gamma_1u_4}{\beta_2}\left(\bar{\bar{u}}_3 - \frac{\gamma}{\gamma_2}\right)$$

Then, $\frac{dw_6}{dt} < 0$ under conditions (31) and (51). Hence, W_6 is Lyapunov function, and thus, F_6 is globally stable in R_+^4.

Theorem 8 Assume the local stability condition (34) and that the following;

$$\alpha_1 \tilde{u}_1 + r_2 < \alpha, \tag{52}$$

is satisfied, then $F_7 = (\tilde{u}_1, 0, \tilde{u}_3, 0)$ is globally stable.

Proof: Define $W_7 = c_1\left(u_1 - \tilde{u}_1 - \tilde{u}_1 \ln\frac{u_1}{\tilde{u}_1}\right) + c_2 u_2 + c_3\left(u_3 - \tilde{u}_3 - \tilde{u}_3\, ln\frac{u_3}{\tilde{u}_3}\right) + c_4 u_4$. Then,

$$\frac{dw_7}{dt} = \mathrm{c}_1(u_1 - \tilde{u}_1)\left(-\frac{r_1(u_1 - \tilde{u}_1)}{k} - \alpha_1 u_2 - \beta_1(u_3 - \tilde{u}_3)\right) + \mathrm{c}_2 u_2\left(r_2\left(1 - \frac{u_2}{l}\right) - \alpha_2 u_1 - \alpha\right)$$

$$+ c_3(u_3 - \tilde{u}_3)\left(-\frac{r_3(u_3 - \tilde{u}_3)}{m} + \beta_2(u_1 - \tilde{u}_1) - \gamma_1 u_4\right) + c_4 u_4(\gamma_2 u_3 - \gamma)$$

Further,

$$\frac{\mathrm{dw}_7}{\mathrm{dt}} = -\frac{c_1 r_1}{k} - (c_1\beta_1 - c_3\beta_2)(u_1 - \tilde{u}_1)(u_3 - \tilde{u}_3) - \frac{c_3 r_3}{m}(u_3 - \tilde{u}_3)^2 - c_1\alpha_1 u_1 u_2 + c_1\alpha_1\tilde{u}_1 u_2$$

$$+ c_2 u_2 r_2 - \frac{c_2 r_2 u_2^2}{l} - \alpha_2 c_2 u_1 u_2 - \alpha c_2 u_2 - (c_3\gamma_1 - c_4\gamma_2)u_3 u_4 - \gamma c_4 u_4$$

By choosing the positive constants as: $c_1 = c_2 = 1, c_3 = \frac{\beta_1}{\beta_2}, c_4 = \frac{\gamma_1\beta_1}{\gamma_2\beta_2}$, the following is obtained

$$\frac{dw_7}{dt} = -\frac{r_1(u_1 - \tilde{u}_1)^2}{k} + u_2(\alpha_1\tilde{u}_1 + r_2 - \alpha) - \frac{r_2 u_2^2}{l} - (\alpha_1 + \alpha_2)u_1 u_2 - \frac{r_3\beta_1}{m\beta_2}(u_3 - \tilde{u}_3)^2$$

$$+ \frac{\gamma_1\beta_1 u_4}{\beta_2}\left(\tilde{u}_3 - \frac{\gamma}{\gamma_2}\right)$$

Then, $\frac{dw_7}{dt} < 0$ under conditions (34) and (52). Hence, W_7 is Lyapunov function and therefore, F_7 is globally sable in R_+^4.

Theorem 9 Assume that the local stability condition (36) holds along with the following:

$$kl(\alpha_1 + \alpha_2)^2 \leq 4 r_1 r_2. \tag{53}$$

Then, $F_8 = (u_1^\circ, u_2^\circ, 0{,}0)$ is global stable in R_+^4.

Proof: Define $W_8 = c_1\left(u_1 - u_1^\circ - u_1^\circ \ln\frac{u_1}{u_1^\circ}\right) + c_2\left(u_2 - u_2^\circ - u_2^\circ \ln\frac{u_2}{u_2^\circ}\right) + c_3 u_3 + c_4 u_4$, then:

$$\frac{dw_8}{dt} = c_1(u_1 - u_1^\circ)\left(-\frac{r_1(u_1 - u_1^\circ)}{k} - \alpha_1(u_2 - u_2^\circ) - \beta_1 u_3\right)$$
$$+ c_2(u_2 - u_2^\circ)\left(-\frac{r_2(u_2 - u_2^\circ)}{l} - \alpha_2(u_1 - u_1^\circ)\right)$$
$$+ c_3\left(r_3 - \frac{r_3 u_3}{m} + \beta_2 u_1 - \beta_0 - \gamma_1 u_4\right) + c_4 u_4(\gamma_2 u_3 - \gamma)$$

i.e.,

$$\frac{dw_8}{dt} = -\frac{c_1 r_1}{k}(u_1 - u_1^\circ)^2 - (c_1\alpha_1 + c_2\alpha_2)(u_1 - u_1^\circ)(u_2 - u_2^\circ) - \frac{c_2 r_2 (u_2 - u_2^\circ)^2}{l}$$
$$+ c_1\beta_1 u_1^\circ u_3 + c_3 r_3 u_3 - \frac{c_3 r_3 u_3^2}{m} + (c_3\beta_2 - c_1\beta_1)u_1 u_3 - c_3\beta_0 u_3$$
$$+ (c_4\gamma_2 - c_3\gamma_1)u_3 u_4 - c_4\gamma u_4$$

By choosing the positive constants as: $c_1 = c_2 = 1, c_3 = \frac{\beta_1}{\beta_2}, c_4 = \frac{\beta_1\gamma_1}{\beta_2\gamma_2}$, the following is obtained: $\frac{dw_8}{dt} = -\left(\frac{r_1(u_1 - u_1^\circ)^2}{k} + (\alpha_1 + \alpha_2)(u_1 - u_1^\circ)(u_2 - u_2^\circ) + \frac{r_2(u_2 - u_2^\circ)^2}{l}\right) - \frac{\beta_1}{\beta_2}\frac{r_3 u_3^2}{m} - \frac{\gamma\beta_1\gamma_1}{\beta_2\gamma_2}u_4 + \beta_1 u_3\left(\frac{\beta_2 u_1^\circ + r_3 - \beta_0}{\beta_2}\right)$, which means that:

$\frac{dw_8}{dt} \le -\left(\sqrt{\frac{r_1}{k}}(u_1 - u_1^\circ)^2 + +\sqrt{\frac{r_2}{l}}(u_2 - u_2^\circ)^2\right)^2 - \frac{\beta_1}{\beta_2}\frac{r_3 u_3^2}{m} - \frac{\gamma\beta_1\gamma_1 u_4}{\beta_2\gamma_2} + \beta_1 u_3\left(\frac{\beta_2 u_1^\circ + r_3 - \beta_0}{\beta_2}\right)$ Thus, $\frac{dw_8}{dt} < 0$ under conditions (36) and (53), and hence, W_8 is Lyapunov function. Therefore, F_8 is global stable in R_+^4.

Theorem 10 Suppose that:

$$\frac{r_1\beta_2}{\beta_1} + \alpha_2 u_2' < \beta_2 u_3'. \tag{54}$$

Then, $F_9 = (0, u_2', u_2' u_3', u_4')$ is globally stable in R_+^4.

Proof: Define: $W_9 = c_1 u_1 + c_2\left(u_2 - u_2' - u_2' \ln\frac{u_2}{u_2'}\right) + c_3\left(u_3 - u_3' - u_3' \ln\frac{u_3}{u_3'}\right) + c_4\left(u_4 - u_4' - u_4' \ln\frac{u_4}{u_4'}\right)$. Then,

$$\frac{dw_9}{dt} = c_1 u_1 \left(r_1 \left(1 - \frac{u_1}{k} \right) - \alpha_1 u_2 - \beta_1 u_3 \right) + c_2 (u_2 - u_2') \left(- \frac{r_2 (u_2 - u_2')}{l} - \alpha_2 u_1 \right)$$
$$+ c_3 (u_3 - u_3') \left(- \frac{r_3 (u_3 - \tilde{u}_3)}{m} + \beta_2 u_1 - \beta_0 - \gamma_1 (u_4 - u_4') \right)$$
$$+ c_4 (u_4 - u_4') \big(\gamma_2 (u_3 - \tilde{u}_3) \big)$$

Therefore,

$$\frac{dw_9}{dt} = c_1 u_1 r_1 - \frac{c_1 u_1^2 r_1}{k} - c_1 \alpha_1 u_1 u_2 + (c_3 \beta_2 - c_1 \beta_1) u_1 u_3 - \frac{r_2}{l} c_2 (u_2 - u_2')^2 - c_2 \alpha_2 u_1 u_2$$
$$+ c_2 \alpha_2 u_1 u_2' - \frac{r_3}{m} \mathrm{c}_3 (\mathrm{u}_3 - u_3')^2 + c_3 \beta_2 u_1 u_3' + (c_4 \gamma_2 - c_3 \gamma_1)(u_3 - u_3')(u_4 - u_4')$$

By choosing the positive constants as: $c_2 = c_3 = 1, c_1 = \frac{\beta_2}{\beta_1}, c_4 = \frac{\gamma_1}{\gamma_2},$ the following is obtained

$$\frac{\mathrm{dw}_9}{\mathrm{dt}} = - \frac{\beta_2 u_1^2 r_1}{\beta_1 k} - \frac{r_2}{l} (u_2 - u_2')^2 - \left(\frac{\beta_2}{\beta_1} \alpha_1 + \alpha_2 \right) u_1 u_2 - \frac{r_3}{m} (u_3 - u_3')^2$$
$$+ u_1 \left(\frac{\beta_2}{\beta_1} r_1 + \alpha_2 u_2' - \beta_2 u_3' \right)$$

Then, $\frac{dw_9}{dt} < 0$ under condition (54), and hence, W_9 is Lyapunov function. Therefore, F_9 is globally stable in R_+^4.

Theorem 11 Assume that:

$$\alpha_1 u_1'' + r_2 < \alpha. \tag{55}$$

Then, $F_{10} = (u_1'', 0, u_3'', u_4'')$ is globally stable in R_+^4.

Proof: Define:

$W_{10} = c_1 \left(u_1 - u_1'' - u_1'' \ln \frac{u_1}{u_1''} \right) + c_2 u_2 + c_3 \left(u_3 - u_3'' - u_3'' \ln \frac{u_3}{u_3''} \right) + c_4 \left(u_4 - u_4'' - u_4'' \ln \frac{u_4}{u_4''} \right)$, thus:

$$\frac{dw_{10}}{dt} = \mathrm{c}_1 (u_1 - u_1'') \left(- \frac{r_1}{k} - \alpha_1 \mathrm{u}_2 - \beta_1 (u_3 - u_3'') \right) + \mathrm{c}_2 u_2 \left(r_2 - \frac{r_2 u_2}{l} - \alpha_2 \mathrm{u}_1 - \alpha \right)$$
$$+ \mathrm{c}_3 (\mathrm{u}_3 - u_3'') \left(- \frac{\mathrm{r}_3}{\mathrm{m}} (\mathrm{u}_3 - u_3'') + \beta_2 (u_1 - u_1'') - \gamma_1 (\mathrm{u}_4 - u_4'') \right)$$
$$+ \mathrm{c}_4 (\mathrm{u}_4 - u_4'') \left(\gamma_2 (\mathrm{u}_3 - u_3'') \right)$$

Therefore,

$$\frac{dw_{10}}{dt} = -\frac{c_1 r_1 (u_1 - u_1^{"})^2}{k} - c_1\alpha_1 u_1 u_2 + c_1\alpha_1 u_1^{"} u_2 - (c_1\beta_1 - c_3\beta_2)(u_1 - u_1^{"})(u_3 - u_3^{"})$$

$$+ c_2 u_2 r_2 - \frac{c_2 u_2^2 r_2}{l} - c_2\alpha_2 u_1 u_2 - c_2 u_2 \alpha - \frac{c_3 r_3 (u_3 - u_3^{"})^2}{m}$$

$$- (c_3\gamma_1 - c_4\gamma_2)(u_3 - u_3')(u_4 - u_4')$$

By choosing the positive constants as: $c_1 = c_2 = 1, c_3 = \frac{\beta_1}{\beta_2}, c_4 = \frac{\gamma_1\beta_1}{\gamma_2\beta_2}$, the following is obtained:

$$\frac{dw_{10}}{dt} = -\frac{r_1(u_1 - u_1^{"})^2}{k} - u_1 u_2(\alpha_1 + \alpha_2) + (\alpha_1 u_1^{"} + r_2 - \alpha) u_2 - \frac{u_2^2 r_2}{l} - \frac{\beta_1 r_3}{\beta_2 m}(u_3 - u_3^{"})^2.$$

Then, $\frac{dw_{10}}{dt} < 0$ under the condition (55) and hence, W_{10} is Lyapunov function. Therefore, F_{10} is globally stable in R_+^4.

Theorem 12 Assume that the local stability condition (46) along with condition (53) are satisfied, then $F_{11} = (\check{u}_1, \check{u}_2, \check{u}_3, 0)$ is globally stable.

Proof: Define $W_{11} = c_1\left(u_1 - \check{u}_1 - \check{u}_1 \ln\frac{u_1}{\check{u}_1}\right) + c_2\left(u_2 - \check{u}_2 - \check{u}_2 \ln\frac{u_2}{\check{u}_2}\right) + c_3\left(u_3 - \check{u}_3 - \check{u}_3 \ln\frac{u_3}{\check{u}_3}\right) + c_4 u_4$, thus:

$$\frac{dw_{11}}{dt} = c_1(u_1 - \check{u}_1)\left(-\frac{r_1}{k}(u_1 - \check{u}_1) - \alpha_1(u_2 - \check{u}_2) - \beta_1(u_3 - \check{u}_3)\right) + c_2(u_2 - \check{u}_2)\left(-\frac{r_2}{l}(u_2 - \check{u}_2) - \alpha_2(u_1 - \check{u}_1)\right) + c_3(u_3 - \check{u}_3)\left(-\frac{r_3(u_3 - \check{u}_3)}{m} + \beta_2(u_1 - \check{u}_1) - \gamma_1 u_4\right) + c_4 u_4(\gamma_2 u_3 - \gamma).$$

Therefore,

$$\frac{dw_{11}}{dt} = -\frac{c_1 r_1(u_1 - \check{u}_1)^2}{k} - (c_1\alpha_1 + c_2\alpha_2)(u_1 - \check{u}_1)(u_2 - \check{u}_2) - \frac{c_2 r_2 (u_2 - \check{u}_2)^2}{l} + (c_3\beta_2 - c_1\beta_1)(u_1 - \check{u}_1)(u_3 - \check{u}_3) - \frac{c_3 r_3 (u_3 - \check{u}_3)^2}{m} + (c_4\gamma_2 - c_3\gamma_1) u_3 u_4 + c_3\gamma_1 \check{u}_3 u_4 - \gamma c_4 u_4$$

By choosing the positive constants as: $c_1 = c_2 = 1, c_3 = \frac{\beta_1}{\beta_2}, c_4 = \frac{\gamma_1\beta_1}{\gamma_2\beta_2}$, the following is obtained

$$\frac{dw_{11}}{dt} = -\left(\frac{r_1(u_1 - \check{u}_1)^2}{k} + (\alpha_1 + \alpha_2)(u_1 - \check{u}_1)(u_2 - \check{u}_2) + \frac{r_2(u_2 - \check{u}_2)^2}{l}\right) - \frac{r_3\beta_1(u_3 - \check{u}_3)^2}{m\beta_2} + \frac{\beta_1\gamma_1 u_4}{\beta_2}\left(\check{u}_3 - \frac{\gamma}{\gamma_2}\right)$$

Thus, $\frac{dw_{11}}{dt} \le -\left(\sqrt{\frac{r_1}{k}}(u_1 - \check{u}_1)^2 + \sqrt{\frac{r_2}{l}}(u_2 - \check{u}_2)^2\right)^2 - \frac{r_3\beta_1(u_3 - \check{u}_3)^2}{m\beta_2} + \frac{\beta_1\gamma_1 u_4}{\beta_2}\left(\check{u}_3 - \frac{\gamma}{\gamma_2}\right).$

Then, $\frac{dw_{11}}{dt} < 0$ under conditions (46) and (53). Therefore, W_{11} is a Lyapunov function, and hence, F_{11} is globally stable in R_+^4.

Theorem 13 Assume that condition (53) is satisfied, then $F_{12} = (u_1^*, u_2^*, u_3^*, u_4^*)$ is globally asymptotically stable.

Proof: Define:

$$W_{12} = c_1\left(u_1^* - u_1^* - u_1^* \, ln\frac{u_1}{u_1^*}\right) + c_2\left(u_2^* - u_2^* - u_2^* \, ln\frac{u_2}{u_2^*}\right) + c_3\left(u_3^* - u_3^* - u_3^* \, ln\frac{u_3}{u_3^*}\right)$$
$$+ c_4\left(u_4 - u_4^* - u_4^* \ln\frac{u_4}{u_4^*}\right)$$

Therefore,

$$\frac{\mathrm{dw}_{12}}{\mathrm{dt}} = -\frac{\mathrm{c}_1 r_1 (u_1 - u_1^*)^2}{k} - (\mathrm{c}_1\alpha_1 + c_2\alpha_2)(u_1 - u_1^*)(u_2 - u_2^*) - \frac{c_2 r_2 (u_2 - u_2^*)^2}{l}$$
$$- (c_1\beta_1 - c_3\beta_2)(u_1 - u_1^*)(u_3 - u_3^*) - \frac{c_3 r_3}{m}(u_3 - u_3^*)^2$$
$$- (c_3\gamma_1 - c_4\gamma_2)(u_3 - u_3^*)(u_4 - u_4^*)$$

By choosing the positive constants as: $c_1 = c_2 = 1, c_3 = \frac{\beta_1}{\beta_2}, c_4 = \frac{\beta_1\gamma_1}{\beta_2\gamma_2}$, the following is obtained:

$$\frac{\mathrm{dw}_{12}}{\mathrm{dt}} = -\frac{r_1}{k}(u_1 - u_1^*)^2 - (\alpha_1 + \alpha_2)(u_1 - u_1^*)(u_2 - u_2^*) - \frac{r_2}{l}(u_2 - u_2^*)^2 - \frac{r_3\beta_1}{m\beta_2}(u_3 - u_3^*)^2$$
$$\frac{\mathrm{dw}_{12}}{\mathrm{dt}} = -\left(\frac{r_1}{k}(u_1 - u_1^*)^2 + (\alpha_1 + \alpha_2)(u_1 - u_1^*)(u_2 - u_2^*) + \frac{r_2}{l}(u_2 - u_2^*)^2\right)$$
$$-\frac{r_3\beta_1}{m\beta_2}(u_3 - u_3^*)^2 \frac{\mathrm{dw}_{12}}{\mathrm{dt}}$$
$$\leq -\left(\sqrt{\frac{r_1}{k}}(u_1 - u_1^*)^2 + \sqrt{\frac{r_2}{l}}(u_2 - u_2^*)^2\right)^2 - \frac{r_3\beta_1}{m\beta_2}(u_3 - u_3^*)^2$$

Then, $\frac{dw_{12}}{dt} \leq 0$ under the condition (53), which is negative semi-definite and thus, F_{12} is Lyapunov stable. However, the set $\delta = \{(u_1, u_2, u_3, u_4): \dot{\mathrm{w}}_{12}(u_1, u_2, u_3, u_4) = 0\}$, which is the set $\delta = \{(u_1, u_2, u_3, u_4): u_1 = u_1^*, u_2 = u_2^*, u_3 = u_3^*\}$, does not include any trajectory of the system, except for $F_{12} = (u_1^*, u_2^*, u_3^*, u_4^*)$. Therefore, by LaSalle's invariance principle F_{12} is a globally stable point in the interior of R_+^4.

7. NUMERICAL ANALYSIS

This section aims to find the system's critical parameters that affect the proposed system's behaviour by using numerical simulations. The dynamics of the model (1) are obtained by solving system (1) numerically using the Predictor-Corrector method, with the sixth-order Runge-Kutta method through the help of MATLAB. Then, the time series of the solutions of system (1) are drawn for the following set of parameters:

$$r_1 = 0.3, r_2 = 0.5, r_3 = 0.4, k = 3.5, l = 4, m = 3, \alpha_1 = 0.03, \alpha_2 = 0.05, \alpha = 0.04, \beta_1 = 0.07, \beta_2 = 0.04, \beta_0 = 0.001, \gamma_1 = 0.05, \gamma_2 = 0.04, \gamma = 0.03. \qquad (56)$$

For different sets of initial values (20,15,10,8), (15,10, 8,5) and (8,6,5,2), the solution of system (1) approach asymptotically to the globally stable point $F_{12} = (u_1^*, u_2^*, u_3^*, u_4^*) = (3.6,2.21,0.75,3.04)$ (see Figure 1).

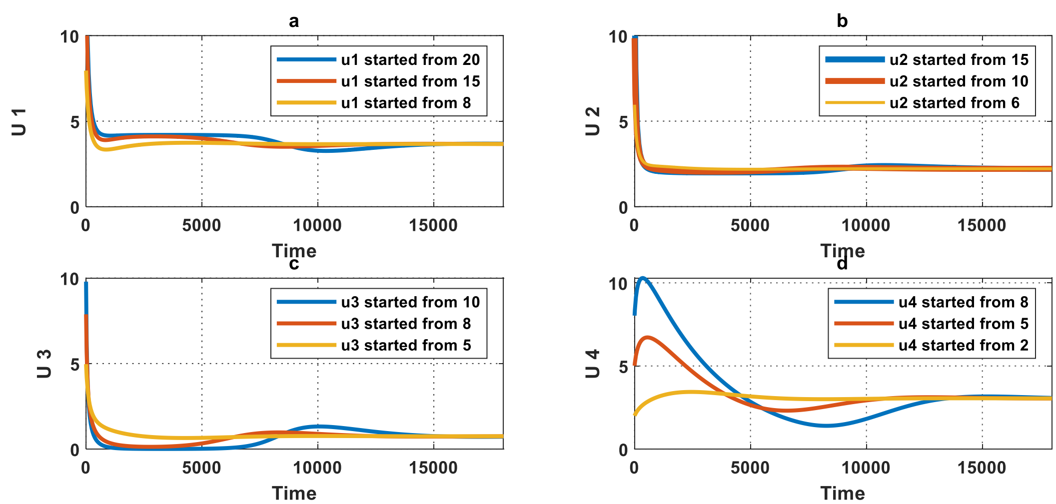

Figure 2 Dynamics of the four species with the data given by (56) with different initial values.

Model (1) is now numerically resolved for the data in (56) to investigate the impact of altering one parameter at a time on system's (1) behaviour. For this purpose, Figure 2 presents the dynamics of the four species with the data given by (56), with different values of α_1. It shows the solution of system (1) approaches its positive equilibrium point F_{12} when $\alpha_1 \leq 0.57$. Furthermore, the second predator becomes zero when $\alpha_1 \geq 0.58$. For example, when $\alpha_1 = 0.58$ the solution, in this case, approaches to $F_{11} = (7.52, 0.67, 0.73, 0)$.

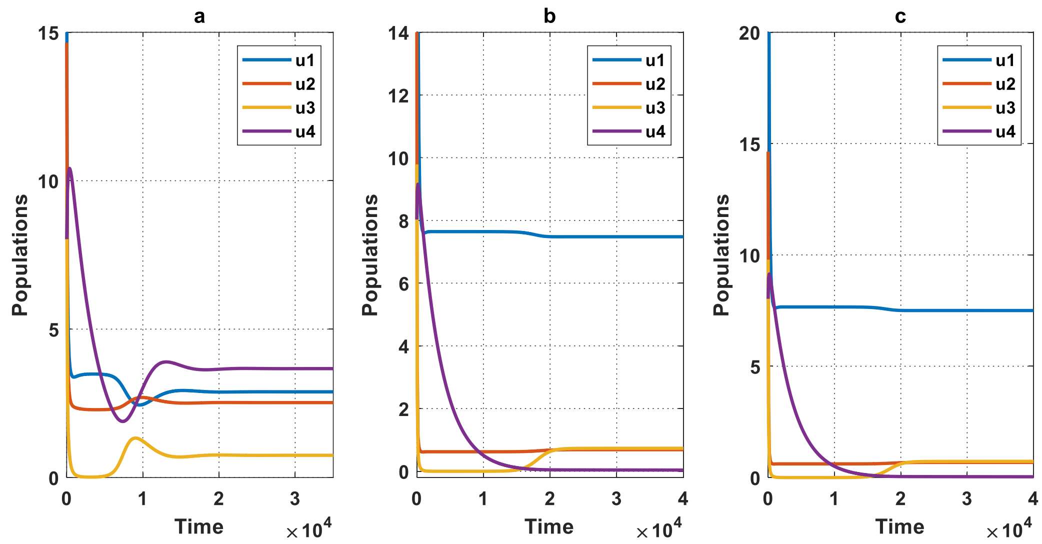

Figure 3 Time series of system (1) for the data given by (56) with: (a) $\alpha_1 = 0.00003$, system (1) converges to (2.8, 2.5, 0.75, 3.66); (b) $\alpha_1 = 0.57$, system (1) converges to (7.47, 0.68, 0.74, 0.005); (c) $\alpha_1 = 0.58$, system (1) converges to (7.52, 0.67, 0.73, 0).

To numerically explore the effect of α_2 the parameters in (56) remain the same except for changing α_2. The solution of system (1) settling down to the interior equilibrium F_{12} for $\alpha_2 \leq 0.15$. Further, the solution of system (1) asymptotically approaches the second prey free equilibrium $F_{10} = (u_1^{"}, 0, u_3^{"}, u_4^{"})$ in the interior of $R^3_{+(u_1 u_3 u_4)}$ for $\alpha_2 \geq 0.16$ (See Figure 4).

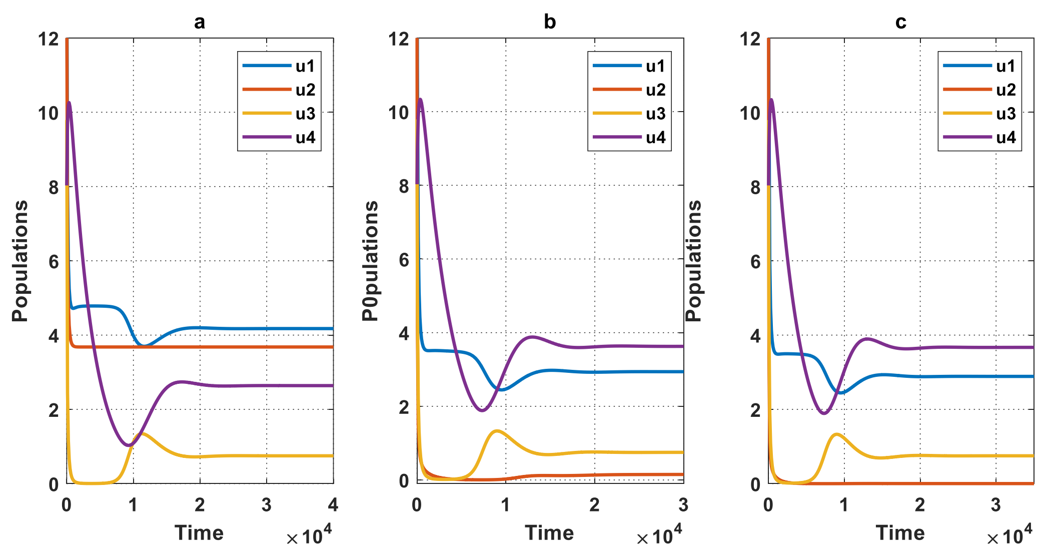

Figure 4 Dynamics of the four species with the data given by Eq. (56) with: (a) $\alpha_2 = 0.00001$, system (1) converges to (4.17, 3.67, 0.75, 2.63); (b) $\alpha_2 = 0.15$, system (1) converges to (2.94, 0.15, 0.75, 3.62;) (c) $\alpha_2 = 0.16$, system (1) converges to (2.88, 0, 0.75, 3.67).

Figure 5 explains system's (1) dynamics with the data given by (56), with different values of the harvesting rate α. It illustrates the solution of system (1) stabilising at F_{12}, when $\alpha \leq o.35$. While the solution of system (1) settles down to F_{10} in Int.$R^3_{+(u_1 u_3 u_4)}$, when $\alpha \geq 0.36$. That means over-harvesting of the second prey harms the survival of this species.

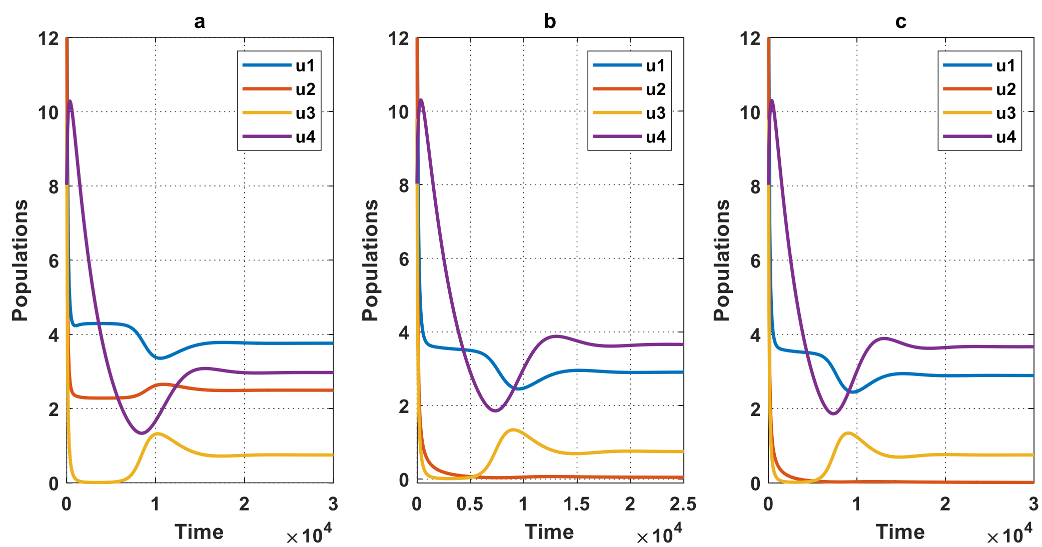

Figure 5 Dynamics of the four species with the data given by (56) with: (a) $\alpha = 0.00001$*, system (1) converges to (3.76, 2.49, 0.75, 2.97); (b)* $\alpha = 0.35$*, system (1) converges to (2.901, 0.03, 0.75, 3.65); (c)* $\alpha = 0.36$*, system (1) converges to (2.88, 0, 0.75, 3.67).*

Now, Figure 6 depicts the system (1) dynamics with (56) at various values of β_1. It demonstrates that when $\beta_1 \leq 0.54$, the solution of system (1) approaches its positive equilibrium point F_{12}. Furthermore, at $\beta_1 \geq 0.55$, the first prey becomes zero and the solution approach asymptotically to $F_9 = (0, u_2', u_3', u_4')$.

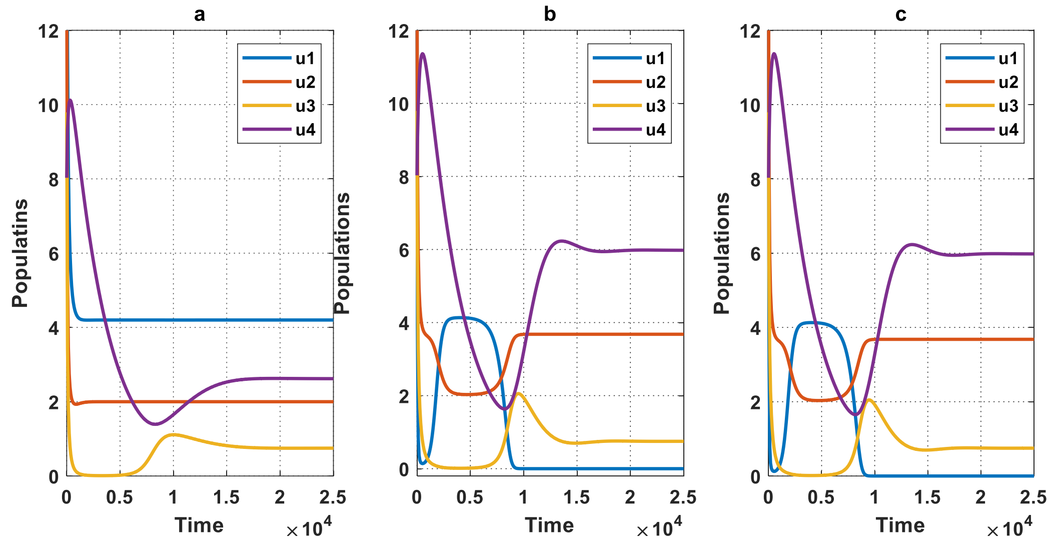

Figure 6 Dynamics of the four species with the data given by (56) with: (a) $\beta_1 = 0.00003$*, system (1) converges to (4.19, 2.000092, 0.75, 2.62); (b)* $\beta_1 = 0.54$*, system (1) converges to (0.05, 3.65, 0.75, 0.93); (c)* $\beta_1 = 0.55$*, system (1) converges to (0, 3.68, 0.75, 5.98).*

Finally, Figure 7 depicts the dynamics of system (1), with the data given by (56) with different values of γ_1. It illustrates the solution of system (1) settles down to F_{12} for different values of γ_1.

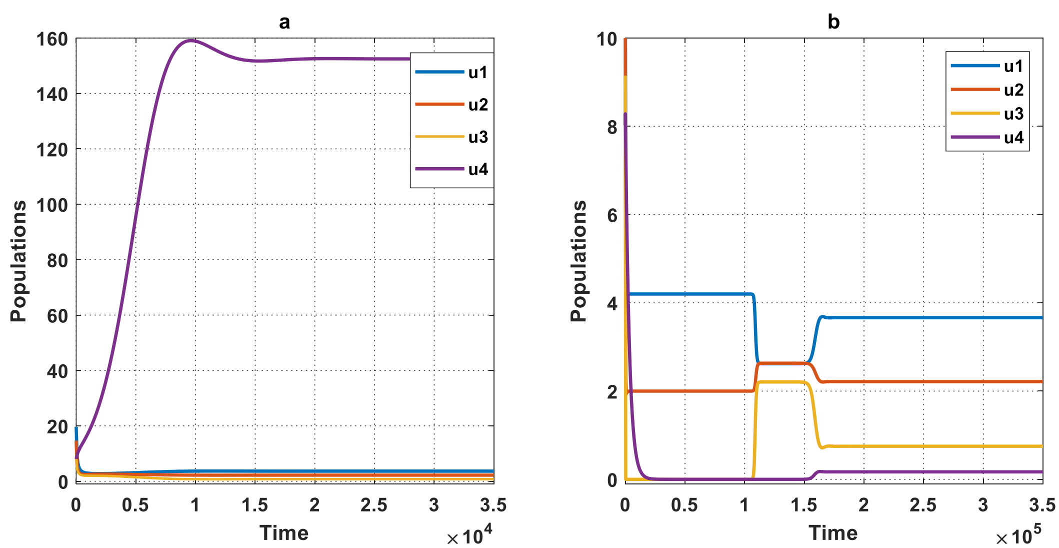

Figure 7 Dynamics of the four species with the data given by (56) with: (a) $\gamma_1 = 0.001$*, system (1) converges to (3.66, 2.21, 0.75, 152.49); (b)* $\gamma_1 = 0.9$*, system (1) converges to (3.66, 2.21, 0.75, 0.16).*

8. CONCLUSION

A four ecological species model consisting of two competitive prey, predator and super predator has been studied. The terms of harvesting the second prey and providing additional food to the first predator have been included. The theoretical analysis of the proposed mathematical model shows the existing conditions of the twelve non-negative equilibrium points. Based on the Routh-Hurwitz stability criteria, the positive equilibria $F_{12} = (u_1^*, u_2^*, u_3^*, u_4^*)$ showed asymptotically stable behaviour under certain conditions. Further, by using the Lyapunov method, the appropriate states that guarantee the global stability of equilibria have been established. According to the numerical simulation results, system movement always happens around the positive equilibria, if the system stability conditions are met. In contrast, an increase in competition rates between the two prey (α_1, α_2), harvesting rate (α) and predation rate (β_1) will lead to the loss of some species.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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