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Abstract. In this paper, some Bayes estimators of the reliability function of Gompertz distribution have been derived based on generalized weighted loss function. In order to get a best understanding of the behaviour of Bayesian estimators, a non-informative prior as well as an informative prior represented by exponential distribution is considered. Monte-Carlo simulation have been employed to compare the performance of different estimates for the reliability function of Gompertz distribution based on Integrated mean squared errors. It was found that Bayes estimators with exponential prior information under the generalized weighted loss function were generally better than the estimators based on Jeffreys prior information.

Introduction

The Gompertz distribution plays an important role in modeling survival times, human mortality and actuarial tables. In reliability and survival studies, many equipment lives are characterized by an increasing hazard rate having Gompertz distribution [1].
The Gompertz distribution was introduced first by Gompertz (1825) to fit mortality tables [2]. It has received a considerable attention from demographers and has been used as a growth model, especially in epidemiological and biomedical studies [3]. Grag *et al.* studied the properties of the Gompertz distribution and obtained maximum likelihood estimators of the parameters [4].
Abu-Zinadah studied different methods of estimation of the shape parameter of the exponentiated Gompertz distribution [5].
The Gompertz distribution is unimodal which has a positive skewness and an increasing hazard rate function. It can be viewed as a truncated extreme value type-I distribution. It can also be interpreted as extensions of the exponential distributions since exponential distributions are limits of sequences of Gompertz distributions [6].
In Applications, Maximum likelihood estimation is often used to estimate the parameters of the Gompertz distribution [7].
In this study we will present a comparison of maximum likelihood estimators and some Bayes estimators of the reliability function of the basic Gompertz distribution. Bayes estimators are derived based on Jefferys non informative prior and exponential prior under the generalized weighted loss function

Model Description

The general Gompertz distribution has a probability density function of the form [3]

$$f(x;\alpha,\beta) = \alpha\, exp\left[\beta x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right] \qquad x \geq 0,\ \alpha > 0,\ \beta > 0$$

The parameters α and β are the shape and the scale parameters respectively.
The corresponding cumulative distribution function is given by

$$F(x;\alpha,\beta) = 1 - exp\left[-\frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right]$$

The Reliability or survival function is

$$R(x;\alpha,\beta) = exp\left[-\frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right]$$

And the hazard rate function is given by

$h(x;\alpha,\beta) = \alpha e^{\beta x}$

An alternative formula for the Gompertz distribution can be derived by assuming the random variable$T = \beta X$. The pdf of the random variable T will take the following form

$f(t;\alpha,\beta) = \frac{\alpha}{\beta}\, exp\left[t - \frac{\alpha}{\beta}(e^t - 1)\right] \quad , t \geq 0$

If we let $\theta = \frac{\alpha}{\beta}$, then the pdf will reduce to the Basic Gompertz distribution with probability density function given by

$f(t\,;\theta) = \theta\exp[t - \theta\,(e^t - 1)] \qquad ; t \geq 0\ , \quad \theta > 0$ (1)

where θ is the shape parameter

The corresponding cumulative distribution function is given by

$F(t;\theta) = 1 - exp[-\theta(e^t - 1)]$ (2)

The Reliability or survival function is

$R(t;\theta) = exp[-\theta(e^t - 1)]$ (3)

And the hazard rate function is given by

$h(t;\theta) = \theta e^t$

Maximum Likelihood Estimator of $R(t;\theta)$

Consider a random sample $T_1, T_2, \ldots, T_n$ from the Basic Gompertz distribution with pdf given by equation (1) . The likelihood function of this sample is

$l(\theta;t) = \theta^n\, exp\ [\sum_{i=1}^{n} t_i - \theta\sum_{i=1}^{n}(e^{t_i} - 1)]$ (4)

The natural log-likelihood function will be:

$\ln l(\theta;t) = nln\theta + \sum_{i=1}^{n} t_i - \theta\sum_{i=1}^{n}(e^{t_i} - 1)$ (5)

Differentiating partially the natural log-likelihood function given by (5), with respect to (θ) and equate to zero, the MLE of θ become:

$\hat{\theta}_{ML} = \frac{n}{\sum_{i=1}^{n}(e^{t_i} - 1)}$ (6)

Now, according to the invariance property of MLE, the MLE of the reliability function of the basic Gompertz distribution is obtained from equation (3) by replacing θ with $\hat{\theta}_{ML}$.

$\hat{R}_{ML}(t;\theta) = exp\left[-\hat{\theta}_{ML}(e^t - 1)\right]; \qquad t \geq 0$ (7)

Bayes Estimators

To obtain Bayes estimators of the reliability function for the basic Gompertz distribution, we considered a non-informative prior as well as an informative prior represented by exponential prior distribution.

i) Posterior Distribution using Jeffreys' Prior Information

With Jefferys' prior information, g (θ) is assumed to be proportional to [8]

$g(\theta) \propto \sqrt{I(\theta)}$

Where $I(\theta) = -nE\left[\frac{\partial^2 \ln\ \ (t;\theta)}{\partial\theta^2}\right]$

is the Fisher information in a single observation about θ.

$ln\, f(t;\theta) = ln\theta + t - \theta(e^t - 1)$

$\frac{\partial ln\, f(t;\theta)}{\partial\theta} = \frac{1}{\theta} - (e^t - 1)$

$\frac{\partial^2 ln\, f(t;\theta)}{\partial\theta^2} = -\frac{1}{\theta^2}$

Therefore

$g_1(\theta) = \frac{b}{\theta}\sqrt{n}$ (8)

where b is a constant

The posterior probability density function of the shape parameter θ can be expressed as:

$$\pi_1(\theta|t) = \frac{l(\boldsymbol{t}|\theta)\ g_1(\theta)}{\int_0^\infty l(\boldsymbol{t}|\theta)\ g_1(\theta)d\theta}$$

$$\pi_1(\theta|\boldsymbol{t}) = \frac{\theta^{n-1}\ exp\ [-\theta\sum_{i=1}^n (\ e^{t_i} - 1)\ ]}{\int_0^\infty \theta^{n-1}\ exp\ [-\theta\sum_{i=1}^n (\ e^{t_i} - 1)\ ]d\theta}$$

$$\pi_1(\theta|\boldsymbol{t}) = \frac{\left[\sum_{i=1}^n (\ e^{t_i}-1)\right]^n}{\Gamma(n)}\theta^{n-1}\ exp\ [-\theta\sum_{i=1}^n (\ e^{t_i} - 1)\ ] \quad (9)$$

It is clear that

$$\theta|t \sim gamma\left(n, \sum_{i=1}^n (\ e^{t_i} - 1)\right)$$

The mean and variance of the posterior distribution are

$$\boldsymbol{E}(\theta|\boldsymbol{t}) = \frac{\boldsymbol{n}}{\sum_{i=1}^n (\ e^{t_i}-1)}$$

$$Var(\theta|\boldsymbol{t}) = \frac{n}{\left[\sum_{i=1}^n (\ e^{t_i} - 1)\right]^2}$$

ii) Posterior Distribution using Exponential Prior

Assuming that θ has an exponential distribution, with pdf

$$g_2(\theta) = \frac{1}{\lambda}e^{-\frac{\theta}{\lambda}},\ \theta > 0,\ \lambda > 0 \quad (10)$$

The posterior probability density function of the shape parameter θcan be derived as follows

$$\pi_2(\theta|\boldsymbol{t}) = \frac{l(\boldsymbol{t}|\theta)\ .\ g_2(\theta)}{\int_0^\infty l(\boldsymbol{t}|\theta)\ .\ g_2(\theta)d\theta}$$

$$\pi_2(\theta|\boldsymbol{t}) = \frac{\theta^n\ exp[\sum_{i=1}^n t_i - \theta\sum_{i=1}^n (\ e^{t_i} - 1)\ ]\ \frac{1}{\lambda}e^{-\frac{\theta}{\lambda}}}{\int_0^\infty \theta^n\ exp[\sum_{i=1}^n t_i - \theta\sum_{i=1}^n (\ e^{t_i} - 1)\ ]\ \frac{1}{\lambda}e^{-\frac{\theta}{\lambda}}d\theta}$$

Let

$$p = \frac{1}{\lambda} + \sum_{i=1}^n (\ e^{t_i} - 1)$$

The posterior distribution simplifies to

$$\pi_2(\theta|\boldsymbol{t}) = \frac{\theta^n e^{-\theta p}}{\int_0^\infty \theta^n e^{-\theta p} d\theta}$$

Using Gamma distribution properties, yields

$$\pi_2(\theta|\boldsymbol{t}) = \frac{p^{n+1}}{\Gamma(n+1)}\theta^n e^{-\theta p} \quad (11)$$

Hence

$$\theta|\boldsymbol{t} \sim Gamma\ (n+1, p)$$

The Generalized weighted Loss Function

The generalized weighted loss function was proposed by Rasheed (2015) which is formulated as follows [9]:

$$l(\hat{\theta}\ ; \theta) = \frac{\left(\sum_{j=0}^k a_j\,\theta^j\right)\left(\hat{\theta}_{GW} - \theta\right)^2}{\theta^c} \qquad \theta > 0, \quad (12)$$

Where $\hat{\theta}$ is an estimate of θ based on the generalized weighted loss function, and a_j, *j=1 ,2, ..., k* are constants.

The risk function is the posterior expectation of the loss function $l(\hat{\theta};\ \theta)$with respect to $\pi(\theta|\boldsymbol{t})$. The value of $\hat{\theta}$ that minimizes the posterior risk is obtained by setting its first partial derivative with respect to $\hat{\theta}$ equal to zero. Thus, according to equation (12), Bayes' estimator of θ obtained by minimizing the expected loss is

$$\hat{\theta} = \frac{a_0 E\left(\frac{1}{\theta^{c-1}}|t\right) + a_1 E\left(\frac{1}{\theta^{c-2}}|t\right) + \cdots + a_K E\left(\frac{1}{\theta^{c-(K+1)}}|t\right)}{a_0 E\left(\frac{1}{\theta^{c}}|t\right) + a_1 E\left(\frac{1}{\theta^{c-1}}|t\right) + \cdots + a_K E\left(\frac{1}{\theta^{c-K}}|t\right)} \quad (13)$$

Now, Bayes estimator of the Reliability function $R(t;\theta)$, based on the generalized weighted loss function, denoted by $\hat{R}(t;\theta)$, may be written as :

$$\hat{R}(t;\theta) = \frac{a_0 E\left(\frac{1}{R^{c-1}}|t\right) + a_1 E\left(\frac{1}{R^{c-2}}|t\right) + \cdots + a_K E\left(\frac{1}{R^{c-(K+1)}}|t\right)}{a_0 E\left(\frac{1}{R^{c}}|t\right) + a_1 E\left(\frac{1}{R^{c-1}}|t\right) + \cdots + a_K E\left(\frac{1}{R^{c-K}}|t\right)} \quad (14)$$

i) Posterior Expectation of $\boldsymbol{R(t;\theta)}$ based upon Jrfferys Prior

In order to find Bayes estimators of the reliability function $R(t;\theta)$, based on Jefferys prior under the generalized weighted loss function, from equation (14), we need to derive the posterior expectation $E(R^m|\boldsymbol{t})$ and $E(R^{-m}|\boldsymbol{t})$ with respect to $\pi_1(\theta|\boldsymbol{t})$.

$$E(R^m|\,\boldsymbol{t}) = \int_0^\infty R^m\,\pi_1(\theta|\boldsymbol{t})d\theta$$
$$= \int_0^\infty exp[-m\theta(e^t-1)]\;\pi_1(\theta|\boldsymbol{t})\,d\theta$$
$$= \int_0^\infty \frac{[\sum_{i=1}^n(e^{t_i}-1)]^n}{\Gamma(n)}\theta^{n-1}exp\left[-\theta\left(\sum_{i=1}^n(e^{t_i}-1)+m(e^t-1)\right)\right]d\theta$$

Hence,

$$E(R^m|\boldsymbol{t}) = \left[\frac{\sum_{i=1}^n(e^{t_i}-1)}{\sum_{i=1}^n(e^{t_i}-1)+m(e^t-1)}\right]^n \quad (15)$$

and

$$E(R^{-m}|\boldsymbol{t}) = \int_0^\infty R^{-m}\,\pi_1(\theta|\,\boldsymbol{t})d\theta$$
$$= \int_0^\infty exp[m\theta(e^t-1)]\;\pi_1(\theta|\,\mathbf{t})\,d\theta$$
$$= \int_0^\infty \frac{[\sum_{i=1}^n(e^{t_i}-1)]^n}{\Gamma(n)}\theta^{n-1}exp\left[-\theta\left(\sum_{i=1}^n(e^{t_i}-1)-m(e^t-1)\right)\right]d\theta$$

Hence,

$$E(R^{-m}|\boldsymbol{t}) = \left[\frac{\sum_{i=1}^n(e^{t_i}-1)}{\sum_{i=1}^n(e^{t_i}-1)-m(e^t-1)}\right]^n \quad (16)$$

ii) Posterior Expectation of $\boldsymbol{R(t;\theta)}$ based upon exponential Prior

In order to find Bayes estimators of the reliability function $R(t;\theta)$, based on exponential prior under the generalized weighted loss function, from equation (14), we need to derive the posterior expectation $E(R^m|\boldsymbol{t})$ and $E(R^{-m}|\boldsymbol{t})$ with respect to $\pi_2(\theta|\boldsymbol{t})$.

$$E(R^m|\,\mathbf{t}) = \int_0^\infty R^m\,\pi_2(\theta|\boldsymbol{t})d\theta$$
$$= \int_0^\infty exp[-m\theta(e^t-1)]\;\pi_2(\theta|\mathbf{t})\,d\theta$$
$$= \int_0^\infty exp[-m\theta(e^t-1)]\;\frac{p^{n+1}}{\Gamma(n+1)}\theta^n e^{-\theta p}\,d\theta$$

Therefore,

$$E(R^m|\boldsymbol{t}) = \left[\frac{p}{p+m(e^t-1)}\right]^{n+1} \quad (17)$$

and

$$\mathrm{E}\,(\mathrm{R}^{-m}|\boldsymbol{t}) = \int_0^\infty R^{-m}\,\pi_2(\theta|\boldsymbol{t})d\theta$$
$$= \int_0^\infty exp[m\theta(e^t-1)]\;\pi_2(\theta|\boldsymbol{t})\,d\theta$$
$$= \int_0^\infty exp[m\theta(e^t-1)]\;\frac{p^{n+1}}{\Gamma(n+1)}\theta^{\mathrm{n}}e^{-\theta p}\,d\theta$$

Hence,

$$\mathrm{E}\,(\mathrm{R}^{-m}|\boldsymbol{t}) = \left[\frac{\mathrm{p}}{\mathrm{p}-m(e^t-1)}\right]^n \quad (18)$$

iii) Bayes Estimators of the Reliability Function $\boldsymbol{R}(\boldsymbol{t};\boldsymbol{\theta})$

In order to obtain Bayes estimators of the reliability function, we applied equation (14) for the cases *k=1, 2, 3 and c = 1, 2* as follows

Taking *k=0* and *c=1*, we obtain

$$\hat{R}_{01} = \frac{a_0}{a_0E\left(\frac{1}{R}|\,\mathbf{t}\right)} = [E(R^{-1}|\boldsymbol{t})]^{-1} \quad (19)$$

Substituting equation (16) with *m=1,* we get

$$\hat{R}_{01} = \left[\frac{\sum_{\mathrm{i=1}}^{\mathrm{n}}(\,\mathrm{e}^{\mathrm{t_i}}-1)}{\sum_{\mathrm{i=1}}^{\mathrm{n}}(\,\mathrm{e}^{\mathrm{t_i}}-1)-(e^t-1)}\right]^{-n}$$

Which is the Bayes estimator of *R(t;θ)* under Jefferys prior

And substituting equation (18) with *m=1*, we get

$$\hat{R}_{01} = \left[\frac{\mathrm{p}}{\mathrm{p}-(e^t-1)}\right]^{-n}$$

Which is the Bayes estimator of *R(t;θ)* under exponential prior

In the same manner we obtain the remaining Bayes estimators of *R(t;θ)*

Taking *k=0* and *c=2*, then we have

$$\hat{R}_{02} = \frac{E\left(\frac{1}{R}|\,\mathbf{t}\right)}{E\left(\frac{1}{R^2}|\,\mathbf{t}\right)} \quad (20)$$

Taking *k=0* and *c=3*, then we have

$$\hat{R}_{03} = \frac{E\left(\frac{1}{R^2}|\,\mathbf{t}\right)}{E\left(\frac{1}{R^3}|\,\mathbf{t}\right)} \quad (21)$$

Taking *k=1* and *c=0*, then we have

$$\hat{R}_{10} = \frac{a_0E(R|\,\mathbf{t})+a_1E(R^2|\,\mathbf{t})}{a_0+a_1E(R|\,\mathbf{t})} \quad (22)$$

Taking *k=1*and *c=1*, then we have

$$\hat{R}_{11} = \frac{a_0+a_1E(R|\,\mathbf{t})}{a_0E\left(\frac{1}{R}|\,\mathbf{t}\right)+a_1} \quad (23)$$

Taking *k=1* and *c=2*, then we have

$$\hat{R}_{12} = \frac{a_0E\left(\frac{1}{R}|\,\boldsymbol{t}\right)+a_1}{a_0E\left(\frac{1}{R^2}|\,\boldsymbol{t}\right)+a_1E\left(\frac{1}{R}|\,\boldsymbol{t}\right)} \quad (24)$$

Taking *k=1* and *c=3*, then we have

$$\hat{R}_{13} = \frac{a_0E\left(\frac{1}{R^2}|\,\boldsymbol{t}\right)+a_1E\left(\frac{1}{R}|\,\boldsymbol{t}\right)}{a_0E\left(\frac{1}{R^3}|\,\boldsymbol{t}\right)+a_1E\left(\frac{1}{R^2}|\,\boldsymbol{t}\right)} \quad (25)$$

Taking *k=2* and *c=1*, then we have

$$\hat{R}_{21} = \frac{a_0+a_1E(R|\boldsymbol{t})+a_2E(R^2|\boldsymbol{t})}{a_0E\left(\frac{1}{R}|\boldsymbol{t}\right)+a_1+a_2E(R|\boldsymbol{t})} \quad (26)$$

Taking *k=2* and *c=2*, then we have

$$\hat{R}_{22} = \frac{a_0E\left(\frac{1}{R}|\boldsymbol{t}\right)+a_1+a_2E(R|\boldsymbol{t})}{a_0E\left(\frac{1}{R^2}|\boldsymbol{t}\right)+a_1E\left(\frac{1}{R}|\boldsymbol{t}\right)+a_2} \quad (27)$$

And taking *k=2* and *c=3*, then we have

$$\hat{R}_{23} = \frac{a_0 E\left(\frac{1}{R^2}|t\right) + a_1 E\left(\frac{1}{R}|t\right) + a_2}{a_0 E\left(\frac{1}{R^3}|t\right) + a_1 E\left(\frac{1}{R^2}|t\right) + a_2 E\left(\frac{1}{R}|t\right)} \quad (28)$$

Simulation and Results

In this section we perform extensive Monte-Carlo simulations to compare the performance of the different estimators obtained in the above sections. We generate random samples from the basic Gompertz distribution with sizes *n* = 20, 30, 50 and 100. Three different values of the shape parameter are chosen as *θ = 0.5,1 and 3*.

The values of hyper-parameters associated with exponential prior distribution are chosen to be *λ=0.5, 1* and *2.* Values of the constants of the generalized weighted loss function were chosen as a_0=10, a_1=100 and a_2=50.

The comparisons between the estimates of the reliability function over L= 5000 repetitions were based on the average values from Integrated Mean Square Error (IMSE), where:

$$IMSE\left(\hat{R}(t;\theta)\right) = \frac{1}{L}\sum_{j=1}^{L}\frac{1}{n_t}\sum_{i=1}^{n_t}\left(\hat{R}_j(t_i;\theta) - R(t_i;\theta)\right)^2 \quad (29)$$

where L is the number of sample replications, $n_t = 4$ is the number of times chosen, where $t = 0.1, 0.3, 0.5, 0.7$ and $\hat{R}_j(t_i\,;\theta)$ is the estimates of $R(t;\theta)$ at the j^{th} run and i^{th} time. The simulation results are summarized through the tables (1) to (6).

Table 1: The IMSE of the estimates of reliability function for Gompertz distribution based on Jeffreys' prior when θ = 0.5

Estimates	n =20	n =30	n =50	n =100
$\hat{R}_{ML}$	0.002411	0.001540	0.000912	0.000449
$\hat{R}_{01}$	0.002533	0.001591	0.000930	0.000454
$\hat{R}_{02}$	0.002826	0.001712	0.000972	0.000464
$\hat{R}_{03}$	0.003190	0.001858	0.001022	0.000476
$\hat{R}_{10}$	0.002161	0.001432	0.000873	0.000439
$\hat{R}_{11}$	0.002341	0.001510	0.000901	0.000446
$\hat{R}_{12}$	0.002581	0.001610	0.000937	0.000455
$\hat{R}_{13}$	0.002886	0.001734	0.000980	0.000465
$\hat{R}_{21}$	0.002303	0.001493	0.000895	0.000445
$\hat{R}_{22}$	0.002530	0.001588	0.000929	0.000453
$\hat{R}_{23}$	0.002822	0.001707	0.000970	0.000463

Table 2: The IMSE of the estimates of reliability function for Gompertz distribution based on the Exponential prior when θ = 0.5

Estimates		n =20	n =30	n =50	n =100
$\hat{R}_{ML}$		0.002411	0.001540	0.000912	0.000449
$\hat{R}_{01}$	λ = 0.5	0.002253	0.001476	0.000891	0.000444
	λ = 1	0.002579	0.001617	0.000942	0.000457
	λ =2	0.002776	0.001700	0.000972	0.000464
$\hat{R}_{02}$	λ = 0.5	0.002503	0.001585	0.000931	0.000454
	λ = 1	0.002900	0.001753	0.000990	0.000469
	λ =2	0.003136	0.001851	0.001024	0.000477
$\hat{R}_{03}$	λ = 0.5	0.002814	0.001717	0.000978	0.000466
	λ = 1	0.003290	0.001914	0.001046	0.000482
	λ =2	0.003569	0.002028	0.001085	0.000491
$\hat{R}_{10}$	λ = 0.5	0.001938	0.001334	0.000837	0.000431
	λ = 1	0.002149	0.001429	0.000873	0.000440
	λ =2	0.002284	0.001488	0.000895	0.000445

$\hat{R}_{11}$	λ = 0.5	0.002090	0.001403	0.000864	0.000437
	λ = 1	0.002361	0.001522	0.000908	0.000448
	λ =2	0.002528	0.001594	0.000933	0.000455
$\hat{R}_{12}$	λ = 0.5	0.002293	0.001493	0.000897	0.000446
	λ = 1	0.002630	0.001638	0.000949	0.000459
	λ =2	0.002834	0.001724	0.000979	0.000466
$\hat{R}_{13}$	λ = 0.5	0.002553	0.001605	0.000938	0.000456
	λ = 1	0.002963	0.001778	0.000999	0.000471
	λ =2	0.003207	0.001878	0.001033	0.000479
$\hat{R}_{21}$	λ = 0.5	0.002057	0.001388	0.000858	0.000436
	λ = 1	0.002315	0.001501	0.000900	0.000446
	λ =2	0.002475	0.001571	0.000925	0.000452
$\hat{R}_{22}$	λ = 0.5	0.002249	0.001473	0.000890	0.000444
	λ = 1	0.002572	0.001612	0.000940	0.000456
	λ =2	0.002768	0.001695	0.000969	0.000463
$\hat{R}_{23}$	λ = 0.5	0.002497	0.001581	0.000929	0.000453
	λ = 1	0.002893	0.001747	0.000987	0.000468
	λ =2	0.003128	0.001844	0.001021	0.000476

Table 3: The IMSE of the estimates of reliability function for Gompertz distribution based on Jeffreys' prior when θ=1

Estimates	n=20	n=30	n =50	n =100
$\hat{R}_{ML}$	0.004168	0.002719	0.001641	0.000822
$\hat{R}_{01}$	0.004467	0.002848	0.001688	0.000834
$\hat{R}_{02}$	0.005298	0.003202	0.001815	0.000865
$\hat{R}_{03}$	0.006481	0.003698	0.001987	0.000906
$\hat{R}_{10}$	0.003773	0.002544	0.001574	0.000805
$\hat{R}_{11}$	0.004081	0.002677	0.001625	0.000818
$\hat{R}_{12}$	0.004685	0.002934	0.001717	0.000840
$\hat{R}_{13}$	0.005624	0.003326	0.001855	0.000874
$\hat{R}_{21}$	0.004042	0.002660	0.001618	0.000816
$\hat{R}_{22}$	0.004607	0.002899	0.001704	0.000837
$\hat{R}_{23}$	0.005506	0.003272	0.001835	0.000869

Table 4: The IMSE of the estimates of reliability function forGompertz distribution based on the Exponential prior when θ =

Estimates		n =20	n =30	n =50	n =100
$\hat{R}_{ML}$		0.004168	0.002719	0.001641	0.000822
$\hat{R}_{01}$	λ = 0.5	0.003392	0.002371	0.001511	0.000789
	λ = 1	0.004006	0.002652	0.001619	0.000817
	λ =2	0.004499	0.002870	0.001700	0.000837
$\hat{R}_{02}$	λ = 0.5	0.003775	0.002548	0.001580	0.000807
	λ = 1	0.004732	0.002976	0.001739	0.000847
	λ =2	0.005417	0.003273	0.001846	0.000874
$\hat{R}_{03}$	λ = 0.5	0.004432	0.002844	0.001691	0.000835
	λ = 1	0.005770	0.003430	0.001903	0.000888
	λ =2	0.006668	0.003813	0.002038	0.000921
$\hat{R}_{10}$	λ = 0.5	0.003321	0.002331	0.001489	0.000782
	λ = 1	0.003415	0.002379	0.001512	0.000789
	λ =2	0.003616	0.002471	0.001548	0.000798

$\hat{R}_{11}$	λ = 0.5	0.003293	0.002322	0.001489	0.000783
	λ = 1	0.003671	0.002497	0.001559	0.000801
	λ =2	0.004034	0.002659	0.001620	0.000817
$\hat{R}_{12}$	λ = 0.5	0.003492	0.002415	0.001528	0.000793
	λ = 1	0.004191	0.002730	0.001647	0.000823
	λ =2	0.004735	0.002967	0.001733	0.000845
$\hat{R}_{13}$	λ = 0.5	0.003948	0.002622	0.001606	0.000813
	λ = 1	0.005009	0.003087	0.001777	0.000856
	λ =2	0.005755	0.003406	0.001890	0.000884
$\hat{R}_{21}$	λ = 0.5	0.003306	0.002328	0.001491	0.000783
	λ = 1	0.003638	0.002482	0.001553	0.000800
	λ =2	0.003976	0.002632	0.001609	0.000814
$\hat{R}_{22}$	λ = 0.5	0.003471	0.002405	0.001523	0.000792
	λ = 1	0.004122	0.002698	0.001634	0.000820
	λ =2	0.004640	0.002923	0.001716	0.000841
$\hat{R}_{23}$	λ = 0.5	0.003892	0.002595	0.001596	0.000810
	λ = 1	0.004903	0.003038	0.001758	0.000851
	λ =2	0.005624	0.003345	0.001867	0.000878

Table 5: The IMSE of the estimates of reliability function for Gompertz distribution based on the Jeffreys' prior when θ=3

Estimates	n =20	n =30	n =50	n =100
$\hat{R}_{ML}$	0.003561	0.002347	0.001434	0.000728
$\hat{R}_{01}$	0.003747	0.002429	0.001465	0.000735
$\hat{R}_{02}$	0.004594	0.002824	0.001615	0.000774
$\hat{R}_{03}$	0.005975	0.003498	0.001877	0.000843
$\hat{R}_{10}$	0.003859	0.002489	0.001484	0.000741
$\hat{R}_{11}$	0.003636	0.002380	0.001446	0.000731
$\hat{R}_{12}$	0.004157	0.002614	0.001533	0.000753
$\hat{R}_{13}$	0.005288	0.003152	0.001738	0.000806
$\hat{R}_{21}$	0.003651	0.002388	0.001448	0.000731
$\hat{R}_{22}$	0.004127	0.002600	0.001528	0.000751
$\hat{R}_{23}$	0.005223	0.003121	0.001726	0.000802

Table 6: The IMSE of the estimates of reliability function for Gompertz distribution based on the Exponential prio when θ=3

Estimates		n =20	n =30	n =50	n =100
$\hat{R}_{ML}$		0.003561	0.002347	0.001434	0.000728
$\hat{R}_{01}$	λ = 0.5	0.004641	0.002873	0.001624	0.000777
	λ = 1	0.003010	0.002095	0.001335	0.000702
	λ =2	0.003167	0.002170	0.001368	0.000711
$\hat{R}_{02}$	λ = 0.5	0.003584	0.002363	0.001430	0.000726
	λ = 1	0.003020	0.002101	0.001341	0.000704
	λ =2	0.003745	0.002447	0.001477	0.000740
$\hat{R}_{03}$	λ = 0.5	0.003130	0.002146	0.001350	0.000705
	λ = 1	0.003587	0.002388	0.001458	0.000736
	λ =2	0.004839	0.002994	0.001695	0.000798

$\hat{R}_{10}$	λ = 0.5	0.007857	0.004408	0.002204	0.000928
	λ = 1	0.004433	0.002765	0.001583	0.000766
	λ =2	0.003615	0.002384	0.001445	0.000731
$\hat{R}_{11}$	λ = 0.5	0.005725	0.003391	0.001821	0.000828
	λ = 1	0.003393	0.002278	0.001403	0.000720
	λ =2	0.003188	0.002182	0.001371	0.000712
$\hat{R}_{12}$	λ = 0.5	0.004278	0.002699	0.001559	0.000760
	λ = 1	0.003057	0.002118	0.001346	0.000705
	λ =2	0.003453	0.002303	0.001419	0.000724
$\hat{R}_{13}$	λ = 0.5	0.003490	0.002320	0.001416	0.000722
	λ = 1	0.003344	0.002261	0.001404	0.000721
	λ =2	0.004300	0.002717	0.001581	0.000767
$\hat{R}_{21}$	λ = 0.5	0.005932	0.003488	0.001857	0.000837
	λ = 1	0.003484	0.002320	0.001419	0.000724
	λ =2	0.003221	0.002197	0.001377	0.000713
$\hat{R}_{22}$	λ = 0.5	0.004427	0.002770	0.001586	0.000767
	λ = 1	0.003099	0.002138	0.001353	0.000707
	λ =2	0.003442	0.002298	0.001417	0.000724
$\hat{R}_{23}$	λ = 0.5	0.003589	0.002368	0.001434	0.000727
	λ = 1	0.003346	0.002262	0.001404	0.000721
	λ =2	0.004255	0.002695	0.001572	0.000764

Discussion

The results of simulation are summarized in the following points:

1. When the value of shape parameter *θ=0.5*, it was found that Bayes estimators of $R(t;\theta)$ under generalized weighted loss function of Gompertz distribution based on exponential prior $\hat{R}_{10}$ (k=1, c=0), $\hat{R}_{21}$ (k=2, c=1) and $\hat{R}_{11}$ (k=1, c=1), has the smallest IMSE when the value of the hyper parameter λ=0.5, as shown in table (2).The second best estimators are Bayes estimator based on exponential prior $\hat{R}_{10}$ (k=1, c=0), when the value of the hyper parameter λ=1 as shown in table (2) followed by Bayes estimator based on Jefferys prior $\hat{R}_{10}$ (k=1, c=0), as shown in table (1).
2. When the value of the shape parameter θ=1, it was found that Bayes estimators of $R(t;\theta)$ based on exponential prior $\hat{R}_{11}$ (k=1, c=1), $\hat{R}_{21}$ (k=2, c=1) and $\hat{R}_{10}$ (k=1, c=0), has the smallest IMSE when the value of the hyper parameter λ=0.5, as shown in table (4). The second best estimators are Bayes estimator based on exponential prior $\hat{R}_{01}$ (k=0, c=1) when the value of the hyper parameter λ= 0.5 followed by Bayes estimator based on exponential prior $\hat{R}_{10}$ (k=1, c=0) when the value of the hyper parameter λ=1, as shown in table (4).
3. When the value of the shape parameter θ=3, it was found that Bayes estimators of $R(t;\theta)$ based on exponential prior $\hat{R}_{01}$ (k=0, c=1), $\hat{R}_{02}$ (k=0, c=2) and $\hat{R}_{12}$ (k=1, c=2) has the smallest IMSE when the value of the hyper parameter λ=1 as shown in table (6).The second best estimators are Bayes estimator based on exponential prior $\hat{R}_{03}$ (k=0, c=3) when the value of the hyper parameter λ= 0.5 followed by Bayes estimator based on exponential prior $\hat{R}_{01}$ (k=0, c=1) as shown in table (6). On the other hand, under the same situation (θ=3), the maximum likelihood estimator $\hat{R}_{ML}$ seem to perform better than Bayes estimators based on Jefferys prior as shown in table (5).

Finally, simulation results showed that some Bayes estimators of $R(t;\theta)$ are better than maximum likelihood estimators especially when the shape parameter θ is relatively small. While Bayes estimators based on exponential prior information under the generalized weighted loss function are generally better than those estimators based on Jeffreys prior information.

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