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Bifurcation Analysis of the Role of Good and Bad Bacteria in the Decomposing Toxins in the Intestine with the Impact of Antibiotic and Probiotics Supplement

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Abstract. This study presents a mathematical model describing the interaction of gut bacteria in the participation of probiotics and antibiotics, assuming that some good bacteria become harmful through mutations due to antibiotic exposure. The qualitative analysis exposes twelve equilibrium points, such as a good-bacteria equilibrium, a bad-bacteria equilibrium, and a coexisting endemic equilibrium in which both bacteria exist while being exposed to antibiotics. The theory of the Sotomayor theorem is applied to study the local bifurcation around all possible equilibrium points. It's noticed that the transcritical and saddle-node bifurcation could occur near some of the system's equilibrium points, while pitchfork bifurcation cannot be accrued at any of them.

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INTRODUCTION

Gut flora is just one of the trillion distinct kinds of bacteria living in the human large intestine, forming an ecosystem together. Most of these microbes have some beneficial properties. Beneficial bacteria live in both the small and large intestines. The stomach's acidity prevents the growth of bacteria. Intestinal microbes have several purposes in the body. To give just one example, vitamin K and vitamin B12 are both produced by bacteria in the gut. Keep harmful bacteria from multiplying too much. The large intestine is responsible for detoxification. Remove the indigestible fibre and part of the carbs and sweets from your meals. Bacterial enzymes degrade the starches in plant cell walls [1]. Most of the nutrients in plants would be lost without these bacteria. These help the body break down plant foods like spinach. Some microorganisms in a person's gut are pathogens that can make them sick. Some bacteria are bad for your health, but many kinds of bacteria are good for you. Bacteria in the gut are important for digestion because they help the body break down food and take in nutrients. It is an important part of every living thing [2]. Gut bacteria are vital to human health because they deliver important nutrients, make vitamin K, help digest cellulose, and stimulate angiogenesis and nerve activity in the gut. Due to changes in their makeup caused by changes in the gut ecosystem caused by things like disease, antibiotics, ageing, stress, lifestyle choices, and bad eating habits, they can also be dangerous. Dysbiosis of the gut bacterial communities can lead to long-term health problems, such as autism, inflammatory bowel disease, cancer, and obesity. When we eat, we also feed the microorganisms in our guts. Like us, these bacteria like to eat carbs, proteins and milk sugars. Both people and the bacteria in our guts benefit from this way of eating [3]. New studies have shed light on the collateral damage that antibiotics cause to gut microorganisms. Several drugs have been shown to have rapid and occasionally long-lasting effects on human gut bacteria's taxonomic, genomic, and functional capacities. Broad-spectrum antibiotics limit bacterial diversity while raising and lowering the

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membership of specific native taxa [4]. In the human body, good bacteria already exist naturally as probiotics. Both beneficial and harmful microbes are constantly infecting our bodies. When you are sick, more harmful germs enter your body, tipping the delicate balance of your immune system. When there are fewer bad bacteria, the balance is restored. Supplementing with probiotics is one method to get the good bacteria that aid digestion [4].

On the other hand, the bifurcation theory is a mathematical technique thought to be used to determine a system's oscillatory solutions and stable state. It is useful in understanding the behaviour of nonlinear dynamic systems, such as the presence and disappearance of equilibrium and periodic orbits [5]. This theory has evolved considerably in the literature by using new methods and ideas. Researchers studied many properties, such as coexistence, extinction persistence, stability and bifurcation [6-13]. This paper studies the local bifurcation behaviour at each equilibrium point to understand the system's dynamic behaviour described in the next section.

MATHEMATICAL MODEL

Suppose an ecosystem in the large intestine contains good bacteria $b_1(t)$ and bad bacteria $b_2(t)$ at time t. $c(t)$ is the non-decomposing toxins in the large intestine at time t. $a(t)$ is the concentration of dissolved antibiotics at time t. Under the above assumptions, the following set of ordinary differential equations is obtained:

$$\begin{aligned}
\frac{db_1}{dt}&=r_1 b_1\left[1-\frac{(b_1+\alpha_1 b_2)}{k}\right]+\beta_0 b_1-(\beta_1+\gamma_1)ab_1-\mu_1 b_1=f_1(b_1,b_2,c,a)\\
\frac{db_2}{dt}&=r_2 b_2\left[1-\frac{(b_2+\alpha_2 b_1)}{k}\right]+\beta_1 ab_1-\gamma_2 ab_2-\gamma_0 b_2-\mu_2 b_2=f_2(b_1,b_2,c,a)\\
\frac{dc}{dt}&=(c_0-c)d+q_1 b_2 c-q_2 b_1 c=f_3(b_1,b_2,c,a)\\
\frac{da}{dt}&=\omega-\mu_0 a=f_4(b_1,b_2,c,a).
\end{aligned} \tag{1}$$

The model's (1) parameters are clearly defined in [14] as: r_1 and r_2 represent the growth rates of good and bad bacteria with carrying capacity k; β_0 is the effectiveness rate of Probiotic supplements; β_1 is the transfer rate of good bacteria to harmful bacteria due to mutations of good bacteria exposed to antibiotics; γ_1 and γ_2 are the eliminating rates of good and bad bacteria by an antibiotic; μ_1 and μ_2 are the natural death rates of b_1 and b_2; γ_0 is the elimination rate of harmful bacteria by the immune system; c_0 is the constant intake of non-decomposing toxins in the intestine; d is the natural degradation of non-decomposing toxins in the intestine; q_1 and q_2 are the increased and decreased rates of non-decomposing toxins due to the large amount of harmful and good bacteria, respectively; ω is the concentration rate of antibiotics; μ_0 is the degradation rate of antibiotics. Under the above examination, the schematic sketch of our system is presented in the following figure.

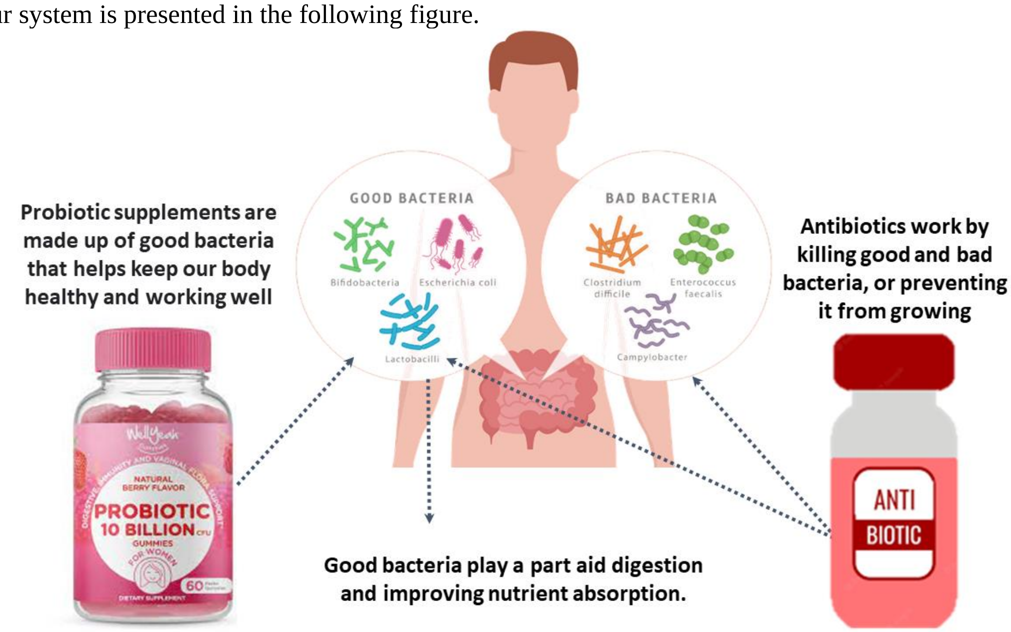

FIGURE 1. The system's (1) schematic sketch.

EQUILIBRIA

System (1) has the following equilibrium points:

1. $s_1 = (\tilde{b}_1, 0,0,0)$.
2. $s_2 = (0, b_2^{;}, 0,0)$.
3. $s_3 = (\ddot{b}_1, 0,0, a^*)$.
4. . $s_4 = (0, b_2^{\dagger}, c^{\dagger}, 0)$.
5. $s_5 = (\dddot{b}_1, 0, \ddot{c}, 0)$.
6. $s_6 = (0, b_2^{''}, 0, a^*)$.
7. $s_7. = (\hat{b}_1, \hat{b}_2, 0,0)$.
8. $s_8 = (\breve{b}_1, 0, \check{c}, a^*)$.
9. . $s_9 = (0, b_2{}', c', a^*)$.
10. $s_{10} = (b_1^-, b_2^-, c^-, 0)$.
11. $s_{11} = (b_1^{=}, b_2^{=}, 0, a^*)$.
12. $s_{12} = (b_1^*, b_2^*, c^*, a^*)$.

The structure, existing conditions and local stability of the above equilibrium points have been clarified [14].
In the following section, we will discuss the possibility of the occurrence of bifurcation near the above steady-states.

LOCAL BIFURCATION ANALYSIS

This section investigates the behaviour of local bifurcations close to all steady-states using Sotomayor's method [15]. System (1) can be reformulated as follows:

$\frac{ds}{dt} = F(s)$, with $s = \begin{pmatrix} b_1 \\ b_2 \\ c \\ a \end{pmatrix}$, and $F = \begin{pmatrix} f_1(b_1, b_2, c, a) \\ f_2(b_1, b_2, c, a) \\ f_3(b_1, b_2, c, a) \\ f_4(b_1, b_2, c, a) \end{pmatrix}$, where $f_i, i = 1,2,3,4$ are the equations on the right-hand side of the system (1). Further, the Jacobian matrix (JM) of system (1) at every point $s = (b_1, b_2, c, a)$ is represented as follows:

$$J = \begin{bmatrix} -\frac{2r_1}{K}\delta_1 - \frac{r_1\alpha_1}{k}\delta_2 - (\beta_1 + \gamma_1)\delta_4 & -\frac{\alpha_1 r_1}{k}\delta_1 & 0 & -(\beta_1 + \gamma_1)\delta_1 \\ -\frac{r_2\alpha_2}{k}\delta_2 + \beta_1\delta_4 & -\frac{r_2\alpha_2}{k}\delta_1 - \frac{2r_2}{k}\delta_2 - \gamma_2\delta_4 & 0 & \beta_1\delta_{1-\gamma_2\delta_2} \\ -q_2\delta_3 & q_1\delta_3 & -q_2\delta_1 + q_1\delta_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For non-zero vector $\delta = (s_1, s_2, s_3, s_4)^T$:

$$D^2(s,s) = \begin{bmatrix} -2\delta_1\left[\frac{r_1\delta_1}{k} + \frac{r_1\alpha_1\delta_2}{k} + (\beta_1 + \gamma_1)\delta_4\right] \\ -\frac{2r_2\alpha_2\delta_1\delta_4}{k} + 2\beta_1\delta_1\delta_4 - \frac{2r_2\delta_2^2}{k} - 2\gamma_2\delta_2\delta_4 \\ 2\delta_3[-q_2\delta_1 + q_1\delta_2] \\ 0 \end{bmatrix}, \quad (2)$$

Further,

$$D^3F(S,S,S) = (0,0,0,0)^T$$

Thus, the pitchfork kind of bifurcation cannot happen when Sotomayor's theorem is applied at $s_i, i = 1,2, \dots ,12$.
The bifurcation near $s_1 = (\tilde{b}_1, 0,0,0)$ is discussed in Theorem 1.

Theorem 1: System (1) at $\mu_2^* = r_2 - \frac{r_2\alpha_2\tilde{b}_1}{k}$, has a transcritical bifurcation around s_1.

***Proof*:** System (1), at s_1, has a zero eigenvalue, say λ_{12}, at $\mu_2^* = r_2 - \frac{r_2\alpha_2\tilde{b}_1}{k}$, and the Jacobian matrix $J^*(s_1) = J(s_1, \mu_2^*)$, becomes:

$$J^*(s_1) = \begin{bmatrix} \frac{-r_1\tilde{b}_1}{k} & \frac{-\alpha_1\tilde{b}_1}{k} & 0 & -(\beta_1+\gamma_1)\tilde{b}_1 \\ 0 & 0 & 0 & \beta_1\tilde{b}_1 \\ 0 & 0 & -d-q_2\tilde{b}_1 & 0 \\ 0 & 0 & 0 & -\mu_0 \end{bmatrix}.$$

Now, let $\delta^{[1]} = \left(\delta_1^{[1]}, \delta_2^{[1]}, \delta_3^{[1]}, \delta_4{}^{[1]}\right)^T$ be the eigenvector corresponding to $\lambda_{12} = 0$. Thus $(J^*(s_1) - \lambda_{12}F)S^{[1]} = 0$, which gives:

$\delta^{[2]} = \left(-\frac{\alpha_1}{r_1}\delta_2^{[1]}, \delta_2^{[1]}, 0{,}0\right)^T$, and $\delta_2^{[1]}$ is any non-zero real number.

Let $U^{[1]} = \left(u_1^{[1]}, u_2^{[1]}, u_3^{[1]}, u_4{}^{[1]}\right)^T$ be the eigenvector associated with λ_{12} of the matrix $J^*(s_1)$. Then $(J_1^{*T} - \lambda_{12}I)U^{[1]} = 0$. By solving this equation for $U^{[1]}$, $U^{[1]} = \left(0, \frac{\mu_0}{\beta_1\tilde{b}_1}u_4^{[1]}, 0, u_4^{[1]}\right)^T$ is obtained, where $u_4^{[1]}$ represent any non-zero real number.

Further,

$$\frac{\partial F}{\partial \mu_2} = F_{\mu_2}(s, \mu_2) = \left(\frac{\partial f_1}{\partial \mu_2}, \frac{\partial f_2}{\partial \mu_2}, \frac{\partial f_3}{\partial \mu_2}, \frac{\partial f_4}{\partial \mu_2}\right)^T = (0, -b_2, 0{,}0)^T$$

So, $F_{\mu_2}(s_1, \mu_2^*) = (0{,}0{,}0{,}0)^T$ and hence $\left(U^{[1]}\right)^T F_{\mu_2}(s_1, \mu_2^*) = 0$.

Further,

$\left(U^{[1]}\right)^T\left[DF_{\mu_2}(s_1, \mu_2^*)\delta^{[1]}\right] = \left(0, \frac{\mu_0}{\beta_1 b_1}u_2^{[1]}, 0, u_4^{[1]}\right)^T \left(-\frac{\alpha_1\delta_2}{r_1}, \delta_2, 0{,}0\right)^T = \frac{\mu_0}{\beta_1 b_1}u_2^{[1]}\,\delta_2 \neq 0$, hence, it is obtained that:

$\left(U^{[1]}\right)^T\left[D^2F_{\mu_2}(s_1, \mu_2^*)\left(\delta^{[1]}, \delta^{[1]}\right)\right] = \frac{-2\delta_2^2 r_2\mu_0}{k\beta_1\tilde{b}_1}u_4^{[1]} \neq 0$, thus, system (1) at s_1 with $\mu_2 = \mu_2{}^*$.

Theorem 2: For $\mu_1^* = r_1 - \frac{r_1\alpha_1\dot{b}_2}{k_1} + \beta_0$, system (1), at s_2 has a transcritical bifurcation.

Proof: System (1), at s_2, has a zero eigenvalue, say λ_{21}, when $\mu_1^* = r_1 - \frac{r_1\alpha_1\dot{b}_2}{k} + \beta_0$, and the Jacobian matrix $J^*(s_2) = J(s_2, \mu_1^*)$, becomes:

$$J^*(s_2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{-r_2\alpha_2\dot{b}_2}{k} & \frac{-r_2\dot{b}_2}{k} & 0 & -\gamma_2\dot{b}_2 \\ 0 & 0 & -d+q_1\dot{b}_2 & 0 \\ 0 & 0 & 0 & -\mu_0 \end{bmatrix}.$$

Now, let $\delta^{[2]} = \left(\delta_1^{[2]}, \delta_2^{[2]}, \delta_3^{[2]}, \delta_4{}^{[2]}\right)^T$ be the eigenvector corresponding to $\lambda_{21} = 0$. Thus $(J^*(I_2) - \lambda_{21}I)\delta^{[4]} = 0$, which gives:

$\delta^{[2]} = \left(\frac{\delta_2^{[2]}}{-\alpha_2}, \delta_2^{[2]}, 0{,}0\right)^T$, and $\delta_2^{[2]}$ is any non-zero real number.

Let $U^{[2]} = \left(u_1^{[2]}, u_2^{[2]}, u_3^{[2]}, u_4{}^{[2]}\right)^T$ be the eigenvector associated with λ_{21} of the matrix $J^*(s_2)$. Then $(J_2^{*T} - \lambda_{21}I)U^{[2]} = 0$. Then, $U^{[2]} = \left(u_1^{[2]}, 0{,}0{,}0\right)^T$, $u_1^{[2]}$ is any non-zero real number.

The following is now taken into account to determine if transcritical bifurcation is possible:

$$\frac{\partial F}{\partial \mu_1} = F_{\mu_1}(s, \mu_1) = \left(\frac{\partial f_1}{\partial \mu_1}, \frac{\partial f_2}{\partial \mu_1}, \frac{\partial f_3}{\partial \mu_1}, \frac{\partial f_4}{\partial \mu_1}\right)^T = (-b_1, 0{,}0{,}0)^T.$$

So, $F_{\mu_1}(s_2, \mu_1^*) = (0{,}0{,}0{,}0)^T$ and hence $\left(U^{[2]}\right)^T F_{\mu_1}(s_2, \mu_1^*) = 0$.

Further,

$\left(U^{[2]}\right)^T\left[DF_{\mu_1}(s_2,\mu_1^*)\delta^{[2]}\right] = \left(u_1^{[2]},0,0,0\right)^T\left(\frac{\delta_2}{-\alpha_2},\delta_2,0,0\right)^T = \frac{\delta_2}{-\alpha_2}u_1^{[2]} \neq 0.$

Hence, it is obtained that:

$\left(U^{[2]}\right)^T\left[D^2F_{\mu_1}(s_2,\mu_1^*)\left(\delta^{[2]},\delta^{[2]}\right)\right] = u_1\left(\frac{-2r_1\delta_1{}^2}{k}\right) \neq 0.$

Thus, system (1) has transcritical bifurcation at s_2 with $\mu_1^* = r_1 - \frac{r_1\alpha_1\dot{b}_2}{k_1} + \beta_0$.

Theorem 3: For $\gamma_0^\circ = r_2 - \frac{r_2\alpha_2\ddot{b}_1}{k} - \gamma_2a^* - \mu_2 - \alpha_1\beta_1a^*$, system (1), at s_3 has a transcritical bifurcation.

Proof: System (1), at s_3, has a zero eigenvalue, say λ_{32} at $\gamma_0^\circ = r_2 - \frac{r_2\alpha_2\ddot{b}_1}{k} - \gamma_2a^* - \mu_2 - \alpha_1\beta_1a^*$ and $J^*(s_3) = J(s_3,\gamma_0^\circ)$, becomes:

$$J^*(s_3) = \begin{bmatrix} -\frac{r_1\ddot{b}_1}{k} & -\frac{r_1\alpha_1\ddot{b}_1}{k} & 0 & -(\beta_1+\gamma_1)\ddot{b}_1 \\ \beta_1a^* & r_2 - \frac{r_2\alpha_2\ddot{b}_1}{k} - \gamma_2a^* - \gamma_0^\circ - \mu_2 & 0 & \beta_1\ddot{b}_1 \\ 0 & 0 & -d - q_2\ddot{b}_1 & 0 \\ 0 & 0 & 0 & -\mu_0 \end{bmatrix}.$$

Now, let $\delta^{[3]} = \left(\delta_1^{[3]},\delta_2^{[3]},\delta_3^{[3]},\delta_4{}^{[3]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{32} = 0$. Thus $(J^*(s_3) - \lambda_{32}I)\delta^{[3]} = 0$, which gives:

$\delta^{[3]} = \left(-\alpha_1\delta_2^{[3]},\delta_2^{[3]},0,0\right)^T$, where $\delta_2^{[3]}$ is any non-zero real number

Let $U^{[3]} = \left(u_1^{[3]},u_2^{[3]},u_3^{[3]},u_4{}^{[3]}\right)^T$ be the eigenvector associated with λ_{23} of the matrix $J^*(s_3)$. Then $(J_3^{*T} - \lambda_{32}I)U^{[3]} = 0$. By solving this equation for $U^{[3]}$, $U^{[3]} = \left(\frac{\beta_1ak}{r_1b_1}u_2^{[3]},u_2^{[3]},0,\frac{-\beta_1[(1+\gamma_1)ak+r_1b_1]}{r_1\mu_0}u_2^{[3]}\right)^T$ is obtained, where $u_2^{[3]}$ represents any non-zero real number.

Now, to check whether the conditions for transcritical bifurcation are met, the following is considered:

$\frac{\partial F}{\partial\gamma_0} = F_{\mu_2}(s,\gamma_0) = \left(\frac{\partial f_1}{\partial\gamma_0},\frac{\partial f_2}{\partial\gamma_0},\frac{\partial f_3}{\partial\gamma_0},\frac{\partial f_4}{\partial\gamma_0}\right)^T = (0,-b_2,0,0)^T.$

So, $F_{\gamma_0}(s_3,\gamma_0^\circ) = (0,0,0,0)^T$ and hence $\left(U^{[3]}\right)^TF_{\gamma_0}(s_3,\gamma_0^\circ) = 0$.

Now,

$\left(U^{[3]}\right)^T\left[DF_{\gamma_0}(s_3,\gamma_0^\circ)\delta^{[3]}\right] = \left(\frac{\beta_1ak}{r_1b_1}u_2^{[3]},u_2^{[3]},0,\frac{-\beta_1[(1+\gamma_1)ak+r_1b_1]}{r_1\mu_0}u_2^{[3]}\right)^T\left(-\alpha_1\delta_2^{[3]},\delta_2^{[3]},0,0\right)^T = -u_2^{[3]}\delta_2^{[3]} \neq 0,$

and hence, it is obtained that:

$$\left(U^{[3]}\right)^T\left[D^2F_{\gamma_0}(s_3,\gamma_0^\circ)\left(\delta^{[3]},\delta^{[3]}\right)\right] = -\frac{2\delta_1^{[3]}\beta_1aku_2^{[3]}}{r_1b_1}\left[\frac{r_1\delta_1^{[3]}}{k} + \frac{r_1\alpha_1\delta_1^{[3]}}{k}\right] - \frac{2r_2u_2^{[3]}\delta_2^{[3]\,2}}{k} \neq 0$$

Thus, system (1) has transcritical bifurcation at s_3 with the parameter $\gamma_0^\circ = r_2 - \frac{r_2\alpha_2\ddot{b}_1}{k} - \gamma_2a^* - \mu_2 - \alpha_1\beta_1a^*$.

Theorem 4: For $\mu_1^* = r_2 - \frac{r_1\alpha_1\widehat{b_2}}{k} + \beta_0$, system (1), at s_4 has a transcritical bifurcation.

Proof: System (1), at s_4, has a zero eigenvalue, say λ_{41}, at $\mu_1^* = r_2 - \frac{r_1\alpha_1\widehat{b_2}}{k} + \beta_0$, and $J^*(s_4) = J(s_4,\mu_1^*)$, becomes:

$$J^*(s_4) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{-r_2\alpha_2\,\widehat{b}_2}{k} & \frac{-r_2\,\widehat{b}_2}{k} & 0 & -\gamma_2\,\widehat{b}_2 \\ -q_1\,\widehat{c} & q_2\,\widehat{c} & -d + q_1b_2^\dagger & 0 \\ 0 & 0 & 0 & -\mu_0 \end{bmatrix}.$$

Now, let $\delta^{[4]} = \left(\delta_1^{[4]},\delta_2^{[4]},\delta_3^{[4]},\delta_4{}^{[4]}\right)^T$ be the eigenvector corresponding to $\lambda_{41} = 0$. Thus $(J^*(s_4) - \lambda_{41}I)\delta^{[4]} = 0$, which gives:

$\delta^{[4]} = \left(-\alpha_2\delta_2^{[4]}, \delta_2^{[4]}, \frac{-\widehat{c}\ \delta_1(q_1+q_2\alpha_2)}{(d+q_1b_2)}, 0\right)^T$, and $\delta_2^{[4]}$ is any non-zero real number.

Let $U^{[4]} = \left(u_1^{[4]}, u_2^{[4]}, u_3^{[4]}, u_4{}^{[4]}\right)^T$ be the eigenvector associated with λ_{41} of the matrix J_5^{*T}. Then $(J_4^{*T} - \lambda_{41}I)U^{[4]} = 0$. Then $U^{[4]}$, $U^{[4]} = \left(u_1^{[4]}, 0,0,0\right)^T$.

Further,

$$\frac{\partial F}{\partial \mu_1^*} = F_{\mu_1^*}(s,\mu_1) = \left(\frac{\partial f_1}{\partial \mu_1}, \frac{\partial f_2}{\partial \mu_1}, \frac{\partial f_3}{\partial \mu_1}, \frac{\partial f_4}{\partial \mu_1}\right)^T = (-b_1, 0,0,0)^T.$$

So, $F_{\mu_1^*}(s_4, \mu_1^*) = (0,0,0,0)^T$ and hence $\left(U^{[4]}\right)^T F_{\mu_1}(s_4, \mu_1^*) = 0$.

Further,

$\left(U^{[4]}\right)^T \left[DF_{\mu_1}(s_4, \mu_1^*)\delta^{[3]}\right] = \left(u_1^{[4]}, 0,0,0\right)^T \left(-\alpha_2\delta_2^{[4]}, \delta_2^{[4]}, \frac{-\widehat{c}\ \delta_1(q_1+q_2\alpha_2)}{(d+q_1b_2)}, 0\right)^T = -\alpha_2\delta_2^{[4]}u_1^{[4]} \neq 0$, hence, it is obtained that:

$$\left(U^{[4]}\right)^T \left[D^2F_{\mu_1^*}(s_4, \mu_1^*)\left(\delta^{[4]}, \delta^{[4]}\right)\right] = \frac{-2r_1\delta_1 u_1^{[4]}[\delta_1 + \alpha_1\delta_2]}{k} \neq 0.$$

Thus, system (1) has transcritical bifurcation at s_4 with $\mu_1^* = r_2 - \frac{r_1\alpha_1\widehat{b_2}}{k} + \beta_0$.

Theorem 5*:* For $\mu_2^{\ddot{*}*} = r_2 - \frac{r_2\alpha_2 b_1^{''}}{k} - \gamma_0$, system (1), at s_5 has a transcritical bifurcation.

Proof: System (1), at s_5, has a zero eigenvalue, say λ_{52}, at $\mu_2^{\ddot{*}*} = r_2 - \frac{r_2\alpha_2 b_1^{''}}{k} - \gamma_0$, and $J^*(s_5) = J(s_5, \mu_2^{\ddot{*}*})$, becomes:

$$J^*(s_5) = \begin{bmatrix} -\frac{r_1 b_1^{''}}{k} & 0 & 0 & -(\beta_1+\gamma_1)b_1^{''} \\ 0 & 0 & 0 & \beta_1 b_1^{''} \\ 0 & 0 & -d - q_2 b_1^{''} & 0 \\ 0 & 0 & 0 & -\mu_0 \end{bmatrix}.$$

Now, let $\delta^{[5]} = \left(\delta_1^{[5]}, \delta_2^{[5]}, \delta_3^{[5]}, \delta_4{}^{[5]}\right)^T$ be the eigenvector corresponding to $\lambda_{52} = 0$. Thus $(J^*(s_5) - \lambda_{52}I)\delta^{[5]} = 0$, which gives:

$\delta^{[5]} = \left(0, \delta_2^{[5]}, 0,0\right)^T$.

Let $U^{[5]} = \left(u_1^{[5]}, u_2^{[5]}, u_3^{[5]}, u_4{}^{[5]}\right)^T$ be the eigenvector associated with λ_{52} of the matrix J_5^{*T}. Then $(J_5^{*T} - \lambda_{52}I)U^{[5]} = 0$. Then, $U^{[5]} = \left(0, \frac{\mu_0}{\beta_1 b_1^{''}}u_4^{[5]}, 0, u_4{}^{[5]}\right)^T$.

The following is now taken into account to determine if transcritical bifurcation is possible:

$$\frac{\partial F}{\partial \mu_2^{\ddot{*}*}} = F_{\mu_2^*}(s,\mu_2) = \left(\frac{\partial f_1}{\partial \mu_2}, \frac{\partial f_2}{\partial \mu_2}, \frac{\partial f_3}{\partial \mu_2}, \frac{\partial f_4}{\partial \mu_2}\right)^T = (0, -b_2, 0,0)^T.$$

So $F_{\mu_2^{\ddot{*}*}}(s_5, \mu_2^{\ddot{*}*}) = (0,0,0,0)^T$ and hence $\left(U^{[5]}\right)^T F_{\mu_2^{\ddot{*}*}}(s_5, \mu_2^{\ddot{*}*}) = 0$.

Hence,

$\left(U^{[5]}\right)^T \left[DF_{\mu_2}(s_5, \mu_2^{\ddot{*}*})\delta^{[5]}\right] = \left(0, \frac{\mu_0}{\beta_1 b_1}u_4^{[5]}, 0, u_4{}^{[5]}\right)^T (0, \delta_2, 0,0)^T = \frac{\mu_0}{\beta_1 b_1}u_4^{[5]}\delta_2^{[5]} \neq 0$, and

$$\left(U^{[5]}\right)^T \left[D^2F_{\mu_2^{\ddot{*}*}}(s_5, \mu_2^{\ddot{*}*})\left(\delta^{[5]}, \delta^{[5]}\right)\right] = \frac{2q_1\delta_2\delta_3\mu_0 u_4^{[5]}}{\beta_1 b_1^{''}} \neq 0.$$

Thus, system (1) has transcritical bifurcation at s_5 with $\mu_2^{\ddot{*}*} = r_2 - \frac{r_2\alpha_2 b_1^{''}}{k} - \gamma_0$.

Theorem 6: For $\mu_1{}^{\ddot{*}} = r_1 - \frac{\alpha_1 r_1 b_2^{''}}{k} + \beta_0 - (\beta_1+\gamma_1)a^*$, system (1), at s_6 has a transcritical bifurcation:

Proof: System (1), at s_6, has a zero eigenvalue, say λ_{61}, at $\mu_1{}^{\ddot{*}} = r_1 - \frac{\alpha_1 r_1 b_2^{''}}{k} + \beta_0 - (\beta_1+\gamma_1)a^*$, and the Jacobian matrix $J^*(s_6) = J(s_6, \mu_1{}^{\ddot{*}})$, becomes:

$$J^*(s_6)=\begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{-r_2\alpha_2 b_2^{''}}{k}+\beta_1 a^* & \frac{-r_2 b_2^{''}}{k} & 0 & -\gamma_2 b_2^{''} \\ 0 & 0 & -d+q_1 b_2^{''} & 0 \\ 0 & 0 & 0 & -\mu_0 \end{bmatrix}.$$

Now, let $\delta^{[6]}=\left(\delta_1^{[6]},\delta_2^{[6]},\delta_3^{[6]},\delta_4{}^{[6]}\right)^T$ be the eigenvector corresponding to $\lambda_{61}=0$. Thus $(J^*(s_6)-\lambda_{61}I)\delta^{[6]}=0$, which gives:

$\delta^{[6]}=\left(\delta_1^{[6]},-\alpha_2\delta_1^{[6]}+\frac{r_2 b_2\beta_1 a\delta_1^{[6]}}{k},0{,}0\right)^T$. $\delta_1^{[6]}$, is any non-zero real number.

Let $U^{[6]}=\left(u_1^{[6]},u_2^{[6]},u_3^{[6]},u_4{}^{[6]}\right)^T$ be the eigenvector associated with λ_{61} of the matrix J_6^{*T}. Then $(J_6^{*T}-\lambda_{61}I)U^{[6]}=0$. Then, $U^{[6]}$, $U^{[6]}=\left(u_1^{[6]},0{,}0{,}0\right)^T$.

The following is now taken into account to determine if transcritical bifurcation is possible:

$$\frac{\partial F}{\partial\mu_1{}^{\ddot{*}}}=F_{\mu_1{}^{\ddot{*}}}\left(s_6,\mu_1{}^{\ddot{*}}\right)=\left(\frac{\partial f_1}{\partial\mu_1},\frac{\partial f_2}{\partial\mu_1},\frac{\partial f_3}{\partial\mu_1},\frac{\partial f_4}{\partial\mu_1}\right)^T=(-b_1,0{,}0{,}0)^T.$$

So, $F_{\mu_2{}^{\ddot{*}}}=\left(s_6,\mu_1{}^{\ddot{*}}\right)=(0{,}0{,}0{,}0)^T$ and hence $\left(U^{[6]}\right)^T F_{\mu_2}\left(s_6,\mu_1{}^{\ddot{*}}\right)=0$.

Hence, the transcritical bifurcation meets its first criterion. Further,

$\left(U^{[6]}\right)^T\left[DF_{\mu_2}\left(s_5,\mu_2^{\ddot{*}*}\right)\delta^{[6]}\right]=\left(u_1^{[6]},0{,}0{,}0\right)^T\left(\delta_1^{[6]},-\alpha_2\delta_1^{[6]}+\frac{r_2 b_2\beta_1 a\delta_1^{[6]}}{k},0{,}0\right)^T=-u_1^{[6]}\delta_1^{[6]}\neq 0.$

Hence,

$$\left(U^{[6]}\right)^T\left[D^2F_{\mu_1}\left(s_6,\mu_1{}^{\ddot{*}}\right)\left(\delta^{[6]},\delta^{[6]}\right)\right]=-2\left(\frac{r_1 u_1^{[6]}\delta_1^2}{k}+\frac{r_1 u_1^{[6]}\alpha_1\delta_1\delta_2}{\mu_0}\right)\neq 0.$$

Therefore, transcritical bifurcation requirements are met. Thus, system (1) has transcritical bifurcation at s_6 with $\mu_1{}^{\ddot{*}}=r_1-\frac{\alpha_1 r_1 b_2^{''}}{k}+\beta_0-(\beta_1+\gamma_1)a^*$.

Theorem 7: For $d^*=q_1\hat{b}_2-q_2\hat{b}_1$, system (1), at s_7 has a transcritical bifurcation if

$$\left(U^{[7]}\right)^T\left[D^2F_d(s_7,d^*)\left(\delta^{[7]},\delta^{[7]}\right)\right]\neq 0. \qquad (3)$$

Proof: System (1), at s_7, has a zero eigenvalue, say λ_{73}, at $d^*=q_1\hat{b}_2-q_2\hat{b}_1$, and the Jacobian matrix $J^*(s_7)=J(s_7,d^*)$, becomes:

$$J^*(s_7)=\begin{bmatrix} -\frac{r_1\hat{b}_1}{k} & -\frac{r_1\alpha_1\hat{b}_1}{k} & 0 & -(\beta_1+\gamma_1)\hat{b}_1 \\ -\frac{r_2\alpha_2\hat{b}_2}{k} & -\frac{r_2\hat{b}_2}{k} & 0 & \beta_1 b_1-\gamma_2\hat{b}_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mu_0 \end{bmatrix}.$$

Now, let $\delta^{[7]}=\left(\delta_1^{[7]},\delta_2^{[7]},\delta_3^{[7]},\delta_4{}^{[7]}\right)^T$ be the eigenvector corresponding to $\lambda_{73}=0$. Thus $(J^*(s_7)-\lambda_{73}I)\delta^{[7]}=0$, which gives:

$\delta^{[7]}=\left(-\alpha_1\delta_2^{[7]},\delta_2^{[7]},\delta_3^{[7]},0\right)^T$, $\delta_2^{[7]}$ and $\delta_3^{[7]}$ are any non-zero real numbers.

Let $U^{[7]}=\left(u_1^{[7]},u_2^{[7]},u_3^{[7]},u_4{}^{[7]}\right)^T$ be the eigenvector associated with λ_{73} of the matrix J_7^{*T}. Then $\left(J_7^{*T}-\lambda_{73}I\right)U^{[7]}=0$. Thus, $U^{[7]}=\left(-\alpha_2 u_2{}^{[7]},u_2^{[7]},u_3^{[7]},\frac{[(\beta_1+\gamma_1)\hat{b}_1\alpha_2+(\beta_1\hat{b}_1-\hat{b}_2\gamma_2)]}{\mu_0}u_2{}^{[7]}\right)^T$.

The following is now taken into account to determine if transcritical bifurcation is possible:

$\frac{\partial F}{\partial d}=F_d(s,d)=\left(\frac{\partial f_1}{\partial d},\frac{\partial f_2}{\partial d},\frac{\partial f_3}{\partial d},\frac{\partial f_4}{\partial d}\right)^T=(0{,}0,c_0-c,0)^T.$

So, $F_d(s_7,d^*)=(0{,}0{,}0{,}0)^T$ and hence $\left(U^{[7]}\right)^T F_d(s_7,d^*)=0$.

Now,

$$\left(U^{[7]}\right)^T\left[DF_d(s_5,d^*)\delta^{[7]}\right] = \left(-\alpha_2 u_2{}^{[7]}, u_2^{[7]}, u_3^{[7]}, \frac{\left[(\beta_1+\gamma_1)\hat{b}_1\alpha_2 + (\beta_1\hat{b}_1 - \hat{b}_2\gamma_2)\right]}{\mu_0}u_2{}^{[7]}\right)^T(-\alpha_1\delta_2, \delta_2, \delta_3, 0)^T$$

$$= \alpha_1\alpha_2\delta_2 u_2{}^{[7]} + u_2^{[7]}\delta_2 + u_3^{[7]}\delta_3 \neq 0.$$

Hence, according to condition (3)

$$\left(U^{[7]}\right)^T\left[D^2F_d(s_7,d^*)\left(\delta^{[7]},\delta^{[7]}\right)\right] = 2\alpha_2 u_2{}^{[7]}\left(\frac{r_1\delta_1^2}{k} + \frac{r_1\alpha_1\delta_2\delta_1}{k}\right) - 2u_2{}^{[7]}\left(\frac{r_2\delta_2^2}{k}\right) + 2u_3{}^{[7]}[-q_2\delta_1\delta_3 + q_1\delta_3\delta_2] \neq 0$$

Thus, system (1) has transcritical bifurcation at s_7 with $d^* = q_1\hat{b}_2 - q_2\hat{b}_1$.

Theorem 8: For $\gamma_2{}^{\hat{*}} = \frac{r_2}{a} - \frac{r_2\alpha_2\check{b}_1}{k} - \frac{(\gamma_0-\mu_2)}{a^*} - \alpha_1\beta_1$, system (1), at s_8 has a transcritical bifurcation if

Proof: System (1), at s_8, has a zero eigenvalue, say λ_{82}, at $\gamma_2{}^{\hat{*}} = \frac{r_2}{a} - \frac{r_2\alpha_2\check{b}_1}{k} - \frac{(\gamma_0-\mu_2)}{a^*} - \alpha_1\beta_1$, and the Jacobian matrix $J^*(s_8) = J(s_8,\mu_2{}^{\hat{*}})$becomes:

$$J^*(s_8) = \begin{bmatrix} -\frac{r_1\check{b}_1}{k} & \frac{-r_1\alpha_1\check{b}_1}{k} & 0 & -(\beta_1+\gamma_1)\check{b}_1 \\ \beta_1 a^* & r_2 - \frac{r_2\alpha_2\check{b}_1}{k} - \gamma_2{}^{\hat{*}}a^* - \gamma_0 - \mu_2 & 0 & \beta_1\check{b}_1 \\ -q_2\check{c} & q_1\check{c} & -d - q_2\check{b}_1 & 0 \\ 0 & 0 & 0 & -\mu_0 \end{bmatrix}.$$

Now, let $\delta^{[8]} = \left(\delta_1^{[8]}, \delta_2^{[8]}, \delta_3^{[8]}, \delta_4{}^{[8]}\right)^T$ be the eigenvector corresponding to $\lambda_{82} = 0$. Thus $(J^*(s_8) - \lambda_{82}I)\delta^{[8]} = 0$, which gives:

$$\delta^{[8]} = \left(-\alpha_1\delta_2^{[8]}, \delta_2^{[8]}, \frac{\delta_2^{[8]}c(-q_2\alpha_1+q_1)}{d+\check{b}_1q_2}, 0\right)^T.$$

Let $U^{[8]} = \left(u_1^{[8]}, u_2^{[8]}, u_3^{[8]}, u_4{}^{[8]}\right)^T$be the eigenvector associated with λ_{82} of the matrix J_8^{*T}. Then $(J_8^{*T} - \lambda_{82}I)U^{[8]} = 0$. By solving this equation for $U^{[8]}$, $U^{[8]} = \left(u_1^{[8]}, \frac{r_1\check{b}_1}{a^*\beta_1 k}u_1^{[8]}, 0, \frac{[-(\beta_1+\gamma_1)\check{b}_1 a k + r_1\check{b}_1^2]}{\mu_0}u_1^{[8]}\right)^T$ is obtained, where $u_1^{[8]}$ is any non-zero real number.

The following is now taken into account to determine if transcritical bifurcation is possible:

$$\frac{\partial F}{\partial \gamma_2{}^{\hat{*}}} = F_{\gamma_2}(s,\gamma_2) = \left(\frac{\partial f_1}{\partial \gamma_2}, \frac{\partial f_2}{\partial \gamma_2}, \frac{\partial f_3}{\partial \gamma_2}, \frac{\partial f_4}{\partial \gamma_2}\right)^T = (0, -ab_2, 0, 0)^T.$$

So, $F_{\gamma_2}\left(s_8,\gamma_2{}^{\hat{*}}\right) = (0,0,0,0)^T$and hence

$$\left(U^{[8]}\right)^T F_{\gamma_2{}^{\hat{*}}}\left(s_8,\gamma_2{}^{\hat{*}}\right) = 0.$$

Now,

$$\left(U^{[8]}\right)^T\left[DF_{\gamma_2}\left(s_8,\gamma_2{}^{\hat{*}}\right)\delta^{[8]}\right] = \left(u_1^{[8]}, \frac{r_1b_1}{a\beta_1 k}u_1^{[8]}, 0, \frac{[-(\beta_1+\gamma_1)\check{b}_1 a^* k + r_1\check{b}_1^2]}{\mu_0}u_1^{[8]}\right)^T\left(-\alpha_1\delta_2^{[8]}, \delta_2^{[8]}, \frac{\delta_2^{[8]}c(-q_2\alpha_1+q_1)}{d+\check{b}_1q_2}, 0\right)^T = -\frac{r_1b_1}{a\beta_1 k}u_1^{[8]}\delta_2^{[8]} - \neq 0.$$

Hence, it is obtained that:

$$\left(U^{[8]}\right)^T\left[D^2F_{\gamma_2}(s_8,\gamma_2)\left(\delta^{[8]},\delta^{[8]}\right)\right] = -2u_1^{[8]}\delta_1^{[8]}\left[\frac{r_1\delta_1^{[8]}}{k} + \frac{r_1\alpha_1\delta_2^{[8]}}{k}\right] - \frac{2\check{b}_1 r_1 r_2 u_1^{[8]}\delta_2^{[8]2}}{\beta_1 a^* k^2} \neq 0.$$

Thus, system (1) has transcritical bifurcation at s_8 with $\gamma_2{}^{\hat{*}}$.

Theorem 9: For $\mu_1{}^{\check{*}} = r_1 - \frac{r_1\alpha_1 b_2{}'}{k} + \beta_0 - (\beta_1+\alpha_1)a$, system (1), at s_9 has a transcritical bifurcation.

Proof: System (1), at s_9, has a zero eigenvalue, say $\lambda_{9\,1}$, at $\mu_1{}^{\check{*}} = r_1 - \frac{r_1\alpha_1 b_2{}'}{k} + \beta_0 - (\beta_1+\alpha_1)a^*$, and the Jacobian matrix $J^*(s_9) = J(s_9,\mu_2{}^{\check{*}})$becomes:

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$$J^*(s_9)=\begin{bmatrix}0 & 0 & 0 & 0\\ \frac{-r_2\alpha_2 b_2{}'}{k}+\beta_1 a^* & \frac{-r_2 b_2{}'}{k_1} & 0 & -\gamma_2 b_2{}'\\ -q_2 c' & q_1 c' & -\frac{c_0}{c'} & 0\\ 0 & 0 & 0 & -\mu_0\end{bmatrix}.$$

Now, let $\delta^{[9]}=\left(\delta_1^{[9]},\delta_2^{[9]},\delta_3^{[9]},\delta_4{}^{[9]}\right)^T$ be the eigenvector corresponding to $\lambda_{9\,1}=0$. Thus $(J^*(s_9)-\lambda_{9\,1}I)\delta^{[9]}=0$, which gives:

$\delta^{[9]}=\left(\delta_1^{[9]},(-\alpha_2+\beta_1 a^*)\delta_1^{[9]},\delta_1^{[9]}\frac{c'^2[-q_2+q_1(-\alpha_2+\beta_1 a^*)]}{c_0},0\right)^T$, and $\delta_1^{[9]}$ is any non-zero real number. Let $U^{[9]}=\left(u_1^{[9]},u_2^{[9]},u_3^{[9]},u_4{}^{[9]}\right)^T$ be the eigenvector associated with $\lambda_{9\,1}$ of the matrix $J^*(s_9)$. Then $(J^*(s_9)-\lambda_{91}I)U^{[9]}=0$.

Then, $U^{[9]}=\left(u_1^{[9]},0,0,0\right)^T$.

The following is now taken into account to determine if transcritical bifurcation is possible:

$$\frac{\partial F}{\partial\mu_2}=F_{\mu_2}(s,\mu_2)=\left(\frac{\partial f_1}{\partial\mu_2},\frac{\partial f_2}{\partial\mu_2},\frac{\partial f_3}{\partial\mu_2},\frac{\partial f_4}{\partial\mu_2}\right)^T=(-b_1,0,0,0)^T.$$

So, $F_{\mu_2}\left(s_9,{\mu_2}^{\check{*}}\right)=(0,0,0,0)^T$ and hence $\left(U^{[9]}\right)^T F_{\mu_2}\left(s_9,{\mu_2}^{\check{*}}\right)=0$.

Now,

$$\left(U^{[9]}\right)^T\left[DF_{\mu_2}\left(s_9,{\mu_2}^{\check{*}}\right)\delta^{[8]}\right]=\left(u_1^{[9]},0,0,0\right)^T\left(\delta_1^{[9]},(-\alpha_2+\beta_1 a^*)\delta_1^{[9]},\delta_1^{[9]}\frac{c'^2[-q_2+q_1(-\alpha_2+\beta_1 a^*)]}{c_0},0\right)^T$$
$$=-u_1^{[9]}\delta_1^{[9]}\neq 0$$

Hence, it is obtained that:

$$\left(U^{[9]}\right)\left[D^2F_{{\mu_2}^{\check{*}}}\left(s_9,{\mu_2}^{\check{*}}\right)\left(\delta^{[9]},\delta^{[9]}\right)\right]=u_1^{[9]}\left[-\frac{2\delta_1^2}{k}-\frac{2r_1\alpha_1\delta_1\delta_2}{k}\right]\neq 0.$$

Therefore, system (1) has transcritical bifurcation at s_9 with ${\mu_1}^{\check{*}}$.

Theorem 10: For $\alpha_1^{**}=\frac{1}{\alpha_2}$, system (1), at s_{10} has a saddle-node bifurcation

Proof: System (1), at s_{10}, has a zero eigenvalue, say $\lambda_{10\,1}$, at $\alpha_1^{**}=\frac{1}{\alpha_2}$, and the Jacobian matrix $J^*(s_{10})=J(s_{10},\alpha_1^{**})$ becomes:

$$J^*(s_{10})=\begin{bmatrix}-\frac{r_1 b_1^-}{k} & -\frac{r_1\alpha_1^{**}b_2^-}{k} & 0 & -(\beta_1+\gamma_1)b_1^-\\ \frac{-r_2\alpha_2 b_2^-}{k} & \frac{-r_2 b_2^-}{k} & 0 & \beta_1 b_1^- -\gamma_2 b_2^-\\ -q_2 c^- & q_1 c^- & -\frac{c_0}{c^-} & 0\\ 0 & 0 & 0 & -\mu_0\end{bmatrix}.$$

Now, let $\delta^{[10]}=\left(\delta_1^{[10]},\delta_2^{[10]},\delta_3^{[10]},\delta_4{}^{[10]}\right)^T$ be the eigenvector corresponding to $\lambda_{10\,1}=0$. Thus $(J^*(s_{10})-\lambda_{10\,1}I)\delta^{[10]}=0$, which gives:

$\delta^{[10]}=\left(-\alpha_1\,\delta_2^{[10]},\delta_2^{[10]},\delta_2^{[10]}\frac{c^2(q_2\alpha_1+q_1)}{c_0},0\right)^T$, and $\delta_2^{[10]}$ is any non-zero real number. Let $U^{[10]}=\left(u_1^{[10]},u_2^{[10]},u_3^{[10]},u_4{}^{[10]}\right)^T$ be the eigenvector associated with $\lambda_{10\,1}$ of the matrix $J_{10}^{\#\,T}$. Then $(J_{10}^{*T}-\lambda_{10\,1}I)U^{[10]}=0$. By solving this equation for $U^{[10]}$,

$U^{[10]}=\left(-\frac{r_2 b_2^-}{r_1\alpha_1 b_1}u_2^{[10]},u_2^{[10]},0,\frac{[(\beta_1+\gamma_1)r_2 b_2^-+(\beta_1 b_1^- -\gamma_2 b_2^-)r_1\alpha_1]}{\mu_0}u_2^{[10]}\right)^T$ is obtained where $u_2^{[10]}$ represents any non-zero real number.

The following is now taken into account to determine if saddle-node bifurcation is possible:

$$\frac{\partial F}{\partial\alpha_1}=F_{\alpha_1}(s_{10},\alpha_1)=\left(\frac{\partial f_1}{\partial\alpha_1},\frac{\partial f_2}{\partial\alpha_1},\frac{\partial f_3}{\partial\alpha_1},\frac{\partial f_4}{\partial\alpha_1}\right)^T=\left(\frac{-r_1 b_1^- b_2^-}{k},0,0,0\right)^T.$$

So, $F_{\alpha_1}(s_{10},\alpha_1^{**}) = \left(\frac{-r_1 b_1^- b_2^-}{k}, 0,0,0\right) \neq 0$.
Now,

$$\left(U^{[10]}\right)^T\left[D^2F_{\alpha_1}(s_{10},\alpha_1^{**})\left(\delta^{[10]},\delta^{[10]}\right)\right] = -\frac{r_2 b_2^-}{r_1\alpha_1 b_1^-}u_2^{[10]}\left(\frac{r_1{\delta_1}^{[10]2}}{k} + \frac{r_1\alpha_1\delta_1\delta_2}{k}\right) - \frac{2r_2{\delta_2}^{[10]2}}{k}u_2^{[10]} \neq 0$$

Thus, system (1) has saddle-node bifurcation at s_{10} wit $\alpha_1^{**} = \frac{1}{\alpha_2}$.

Theorem 11: For ${q_1}^{\ddot{*}} = \frac{d+q_2 b_1^=}{b_2^=}$, then system (1), at s_{11} has a transcritical bifurcation if

$$\left(U^{[11]}\right)^T\left[D^2F_{q_1}(s_{11},{q_1}^{\ddot{*}})\left(\delta^{[11]},\delta^{[11]}\right)\right] \neq 0 \tag{4}$$

Proof: System (1), at s_{11}, has a zero eigenvalue, say $\lambda_{11\,3}$ at ${q_1}^{\ddot{*}} = \frac{d+q_2 b_1^=}{b_2^=}$ and the Jacobian matrix $J^*(s_{11}) = J(s_{11},{q_1}^{\ddot{*}})$, becomes:

$$J^*(s_{11}) = \begin{bmatrix} -\frac{r_1 b_1^=}{k} & \frac{-r_1\alpha_1 b_1^=}{k} & 0 & 0 \\ \frac{-r_1\alpha_2 b_2^=}{k} + \beta_1 a^* & \frac{-r_1 b_2^=}{k} - \frac{\beta_1 a^* b_1^=}{b_2} & 0 & \beta_1 b_1^= - \gamma_2 b_2^= \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mu_0 \end{bmatrix}.$$

let $\delta^{[11]} = \left(\delta_1^{[11]},\delta_2^{[11]},\delta_3^{[11]},{\delta_4}^{[11]}\right)^T$ be the eigenvector corresponding to $\lambda_{11\,3} = 0$. Thus $(J^* I_{11} - \lambda_{11\,3} I)\delta^{[11]} = 0$, which gives:

$\delta^{[11]} = \left(-\alpha_1\,\delta_2^{[11]},\delta_2^{[11]},\delta_3^{[11]},0\right)^T$, and $\delta_2^{[11]},\delta_3^{[11]}$ are any non-zero real numbers.

Let $U^{[11]} = \left(u_1^{[11]},u_2^{[11]},u_3^{[11]},{u_4}^{[11]}\right)^T$ be an eigenvector associated with $\lambda_{11\,3}$ of the matrix ${J_{11}^{\#}}^T$. Then $(J^*(s_{11}) - \lambda_{11\,3} I)U^{[11]} = 0$. By solving this equation for $U^{[11]}$, $U^{[11]} = \left(u_1^{[11]},\frac{r_1 b_1^=}{-r_1\alpha_2 b_2^= + \beta_1 a^* k}{u_1}^{[11]},u_3^{[11]},\frac{-(b_1-\gamma_1)b_1^= r_1}{\mu_0(-r_2\alpha_2 + a^* k)}{u_1}^{[11]}\right)^T$ is obtained, where $u_1^{[11]}$,$u_3^{[11]}$ are any non-zero real numbers.

The following is now taken into account to determine if transcritical bifurcation is possible:

$\frac{\partial F}{\partial q_1} = F_{q_1}(s,q_1) = \left(\frac{\partial f_1}{\partial q_1},\frac{\partial f_2}{\partial q_1},\frac{\partial f_3}{\partial q_1},\frac{\partial f_4}{\partial q_1}\right)^T = (0,0,b_2 c,0)^T$.

So, $F_{q_1}(s_{11},{q_1}^{\ddot{*}}) = (0,0,b_2 c,0)^T = (0,0,0,0)$

Therefore, the first condition of the transcritical bifurcation is met. Now,

$$\begin{aligned}\left(U^{[11]}\right)^T&\left[DF_{q_1}(s_{11},{q_1}^{\ddot{*}})\delta^{[11]}\right] \\ &= \left(u_1^{[11]},\frac{r_1 b_1^=}{-r_1\alpha_2 b_2^= + \beta_1 a^* k}{u_1}^{[11]},u_3^{[11]},\frac{-(b_1^= - \gamma_1)b_1^= r_1}{\mu_0(-r_2\alpha_2 + a^* k)}{u_1}^{[11]}\right)^T\left(-\alpha_1\,\delta_2^{[11]},\delta_2^{[11]},\delta_3^{[11]},0\right)^T \\ &= u_3^{[11]}{u_1}^{[11]} + u_3^{[11]}\delta_3^{[11]} \neq 0.\end{aligned}$$

Hence, it is obtained from condition (4):

$$\begin{aligned}\left(U^{[11]}\right)^T&\left[D^2F_{q_1}(s_{11},{q_1}^{\ddot{*}})\left(\delta^{[11]},\delta^{[11]}\right)\right] \\ &= u_1^{[11]}\left(\frac{-2\delta_1^2}{k} - \frac{2r_1\alpha_1\delta_1\delta_2}{k}\right) - \frac{2r_2\delta_2^2 b_1^= u_1^{[11]}}{r_1\alpha_2 b_2^= k + \beta_1 a^* k^2} + 2\delta_3 u_1^{[11]}(-q_2\delta_1 + q_1\delta_2) \neq 0.\end{aligned}$$

Therefore, transcritical bifurcation requirements are met at s_{11} with ${q_1}^{\ddot{*}} = \frac{d+q_2 b_1^=}{b_2^=}$.

Theorem12: For $\alpha_2^* = \frac{1}{\alpha_1} + \frac{\beta_1 a^* b_1^* k}{\alpha_1 r_2 b_2^*} + \frac{\beta_1 a^* k}{r_2 b_2^*}$, Then system (1), at s_{12} has a saddle-node bifurcation if

$$\left(U^{[12]}\right)^T\left[D^2F_{\alpha_2}(s_{12},\alpha_2^*)\left(\delta^{[12]},\delta^{[12]}\right)\right] \neq 0 \tag{3}$$

Proof: System (1), at s_{12}, has a zero eigenvalue, say $\lambda_{12\,2}$ at $\alpha_2^* = \frac{1}{\alpha_1} + \frac{\beta_1 a^* b_1^* k}{\alpha_1 r_2 b_2^*} + \frac{\beta_1 a^* k}{r_2 b_2^*}$, and the Jacobian matrix $J^*(s_{12}) = J(s_{12}, \gamma_2^*)$, becomes

$$J^*(s_{12}) = \begin{bmatrix} -\frac{r_1 b_1^*}{k} & -\frac{r_1 \alpha_1 b_1^*}{k} & 0 & -(\beta_1+\gamma_1)b_1 \\ \frac{-r_2 \alpha_2{}^* b_2^*}{k} + \beta_1 a & \frac{-\beta_1 a^* b_1^*}{b_2} - \frac{r_2 b_2^*}{k} & 0 & \beta_1 b_1^* - \gamma_2 b_2^* \\ -q_2 c^* & q_1 c^* & -\frac{c_0}{c^*} & 0 \\ 0 & 0 & 0 & -\mu_0 \end{bmatrix}.$$

let $\delta^{[12]} = \left(\delta_1^{[12]}, \delta_2^{[12]}, \delta_3^{[12]}, \delta_4{}^{[12]}\right)^T$ be the eigenvector corresponding to $\lambda_{12\,2} = 0$. Thus $(J^*(s_{12}) - \lambda_{12\,2} I)\delta^{[12]} = 0$, which gives:

$\delta^{[12]} = \left(-\alpha_1 \delta_2^{[12]}, \delta_2^{[12]}, \frac{\delta_2^{[12]}\, c^{*2}(q_2\alpha_1 + q_1)}{c_0}, 0\right)^T$, and $\delta_2^{[12]}$ is any non-zero real number.

Let $U^{[12]} = \left(u_1^{[12]}, u_2^{[12]}, u_3^{[12]}, u_4{}^{[12]}\right)^T$ be an eigenvector associated with $\lambda_{11\,2}$ of the matrix $J_{12}^{\#\,T}$. Then $(J_{12}^{*T} - \lambda_{12\,2} I)U^{[12]} = 0$. By solving this equation for $U^{[12]}$,

$U^{[12]} = \left(\frac{r_2 b_2^*}{r_1 b_1^*}(\alpha_2 + 1)\, u_2^{[12]}, u_2^{[12]}, 0, \frac{[(\beta_1+\gamma_1)(\alpha_2+1) r_2 b_2^* + (\beta_1 b_1^* - \gamma_2 b_2^*) r_1]}{\mu_0} u_2^{[12]}\right)^T$

is obtained, where $u_2^{[12]}$ is any non-zero real number.

The following is now taken into account to determine if saddle-node bifurcation is possible:

$$\frac{\partial F}{\partial \alpha_2} = F_{\alpha_2}(s, \alpha_2) = \left(\frac{\partial f_1}{\partial \alpha_2}, \frac{\partial f_2}{\partial \alpha_2}, \frac{\partial f_3}{\partial \alpha_2}, \frac{\partial f_4}{\partial \alpha_2}\right)^T = (0, -a^* b_2^*, 0, 0)^T.$$

So, $F_{\alpha_2}(s_{12}, \alpha_2^*) = (0, -a^* b_2^*, 0, 0)^T \neq 0$. Therefore, the first condition of the saddle-node bifurcation is met.

Hence, according to condition (3)

$$\left(U^{[12]}\right)^T \left[D^2 F_{\alpha_2}(s_{12}, \alpha_2^*)\left(\delta^{[12]}, \delta^{[12]}\right)\right] = -2\, \frac{r_2 b_2^*}{r_1 b_1^*}(\alpha_2 + 1) u_2^{[12]} \left[\frac{r_1 \delta_1^{[12]2}}{k} + \frac{r_1 \alpha_1 \delta_1^{[12]} \delta_2^{[12]}}{k}\right] - u_2^{[12]} \frac{2 r_2 \delta_2^{[12]2}}{k} \neq 0.$$

This means the second condition of saddle-node bifurcation is satisfied at s_{12} with $\alpha_2^* = \frac{1}{\alpha_1} + \frac{\beta_1 a^* b_1^* k}{\alpha_1 r_2 b_2^*} + \frac{\beta_1 a^* k}{r_2 b_2^*}$.

NUMERICAL ANALYSIS AND DISCUSSION

The purpose of numerical simulations is to identify the critical parameters that impact the whole dynamics of the system (1). Model (1) dynamics can be attained by solving system (1) using the Runge-Kutta method via MATLAB and the Runge-Kutta method. After that, the time series of the solutions is drawn in four cases with the following sets of parameters:

$$r_1 = 0.2, r_2 = 0.4, k = 40, \alpha_1 = 0.1, \alpha_2 = 0.1, \delta_1 = 0.16, \delta_2 = 0.16, \beta_0 = 0.14, \beta_1 = 0.016, \gamma_1 = 0.018, \gamma_2 = 0.017, \gamma_0 = 0.18, d = 0.32, c_0 = 4, q_1 = 0.012, q_2 = 0.014, \omega = 0.6,\ \mu_0 = 0.118. \quad (5)$$

The four instances will be considered to realize the system (1) model's behavior and evaluate the outcome of taking probiotic supplementation and antibiotic on the gastrointestinal tract's performance. The consequences of the four instances will then be compared.

- **Case 1**: The dynamics of the system (1) without probiotic supplementation and antibiotic

In this state, we study the dynamics of the system (1) in the absence of probiotic supplementation ($\beta_0 = 0$) and antibiotic ($\omega = 0$). Figure 2 shows the behaviour of the data given in (5) with $\beta_0 = \omega = 0$. It determines the solution settling asymptotically to $s_{10} = (6.26, 17.33, 6.4, 0)$ in $R^3_{+(b_1, b_2, c)}$ for different initial values. It's clear that the intestine's non-decomposing toxins and harmful bacteria are vast. There is a significant risk of gut wall inflammation in this particular scenario. It is also conceivable for harmful bacteria to be transferred from the lumen into the tissue compartment or circulation. Moreover, there is a possibility of developing colon cancer. In the case of the persistent build-up of undissolved toxins in the intestine. See Figure 2.

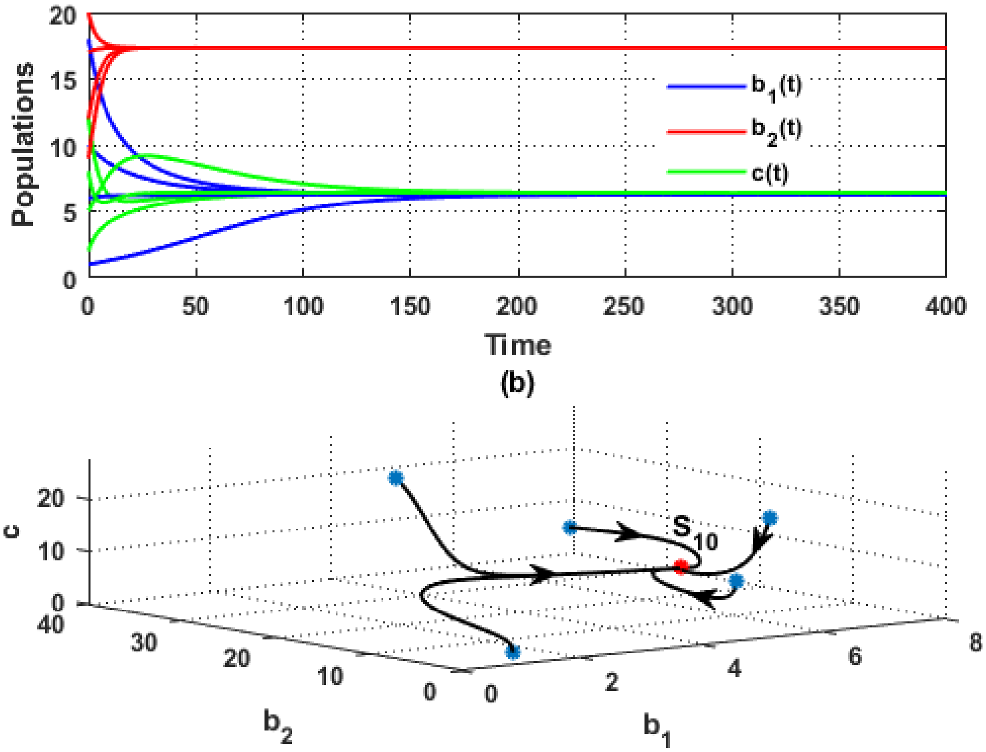

FIGURE 2. The behaviour of the system (1) with the data given by (5) with $\beta_0 = \omega = 0$ and different initial values.

➢ **Case 2**: The dynamics of the system (1) with probiotic supplementation and without antibiotic

In this situation, we study the system's behavior in the present probiotic supplementation ($\beta_0 \neq 0$) and the absence of antibiotics ($\omega = 0$). The simulation shows for different initial values; the solution converges asymptotically to $s_{10} = (34.26, 17.33, 2.16, 0)$ in $R^3_{+(b_1,b_2,c)}$. It's clear from Figure 3 that the intestine's non-decomposing toxins are reduced significantly from 6.4 to 2.16 compared with the previous case. Further, there is a crucial rise in the good bacteria population. In comparison, the population of harmful bacteria is not affected. That means probiotic supplementation has a great effect in decreasing the non-decomposing toxins in the intestine. Consequentially, this lessens the likelihood of accumulated faces and other intestinal waste.

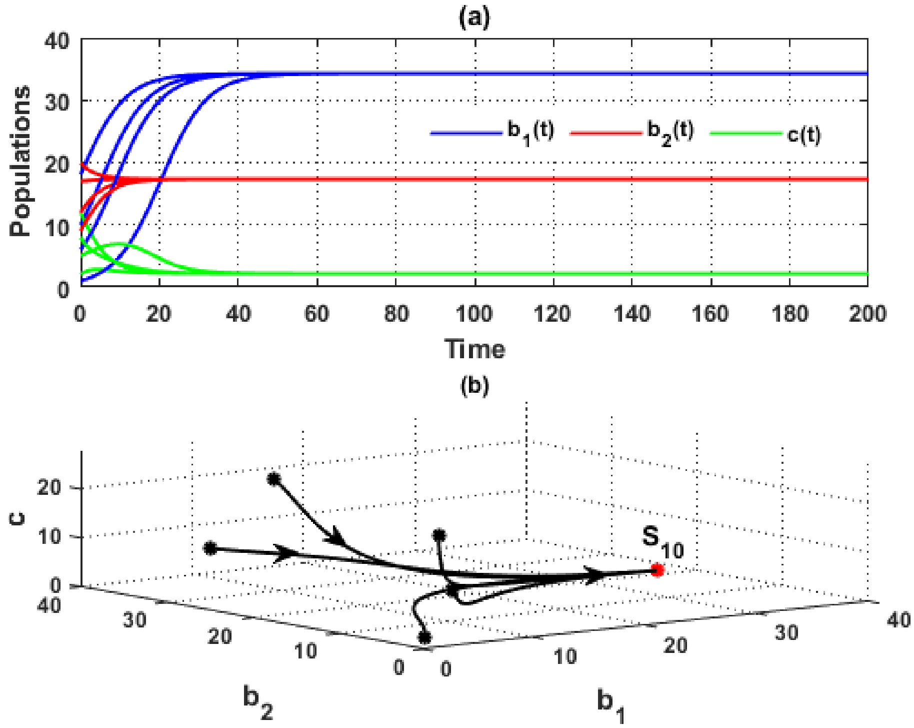

FIGURE 3. The behaviour of the system (1) with the data given by (5) with $\omega = 0$ and different initial values.

➢ **Case 3**: The dynamics of the system (1) with antibiotic and without probiotic supplementation

In this case, we perform the system's behavior in the presence of antibiotics ($\omega \neq 0$) and the absence of probiotic supplementation ($\beta_0 = 0$). The simulation results show for different initial values; the solution converges asymptotically to $s_9 = (0, b_2', c', a^*) = (0,11.57,7.06,5.08)$ in $R^3_{+(b_2,c,a)}$. It is clear from Figure 4 that antibiotics harm both good and harmful bacteria. Comparing this case with case 1, we found that antibiotic has a bigger influence on the proliferation of beneficial bacteria than dangerous bacteria. As a result, this leads to accumulated faces in the intestinal.

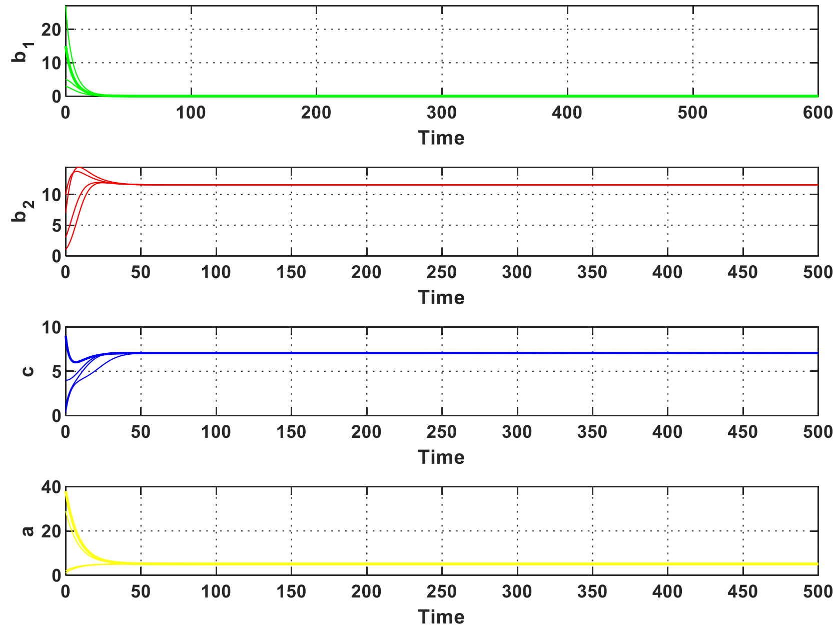

FIGURE 4. The behaviour of the system (1) with $\beta_0 = 0$.

- **Case 4**: The dynamics of the system (1) with probiotic supplementation and antibiotic

In this situation, we examine the dynamics of the system (1) in the presence of probiotic supplementation ($\beta_0 \neq 0$) and antibiotic ($\omega \neq 0$). Figure 5 explains the behaviour of the data given in (5). It demonstrates the solution settling asymptotically to $s_{12} = (4.11, 13.25, 5.81, 5.08)$ in $R^4_{+(b_1,b_2,c,a)}$ for different initial values. The beneficial bacteria stars grow again due to probiotic supplementation compared with case 3. This scenario has a positive effect on reducing the accumulated faces from the intestinal.

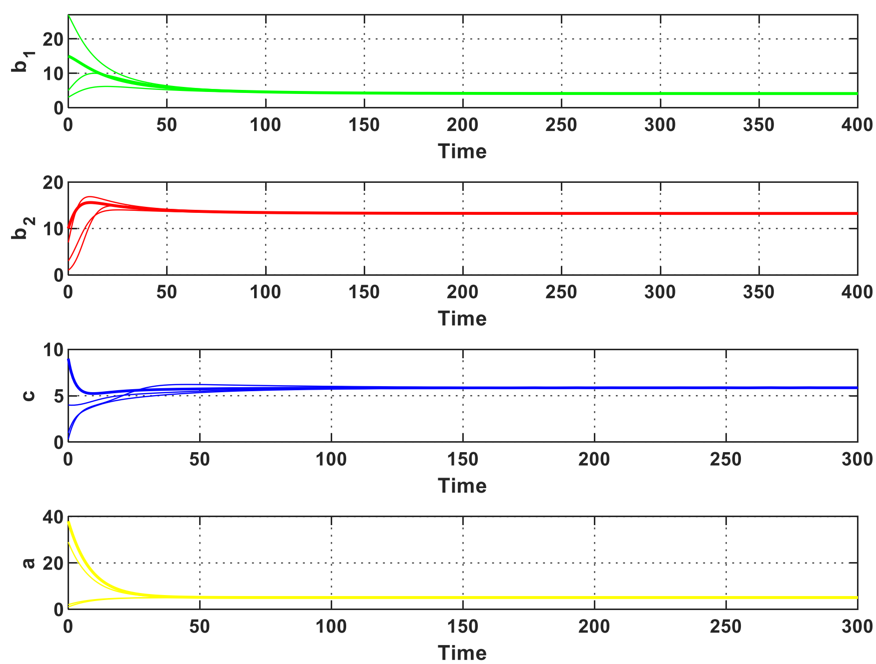

FIGURE 5. The behaviour of the system (1) with the data given by (5).

On the other hand, the impact of increasing the number of probiotic supplementation doses is determined in Figure 6. It is clear that the trajectory of the system (1) converges asymptotically to the positive equilibrium point $s_{12} = (11.85, 15.76, 4.29, 5.08)$ for $\beta_0 = 0.2$. It could be concluded that the beneficial bacteria increased significantly for increasing probiotic supplementation. Further, the amount of accumulated faces decreases slightly.

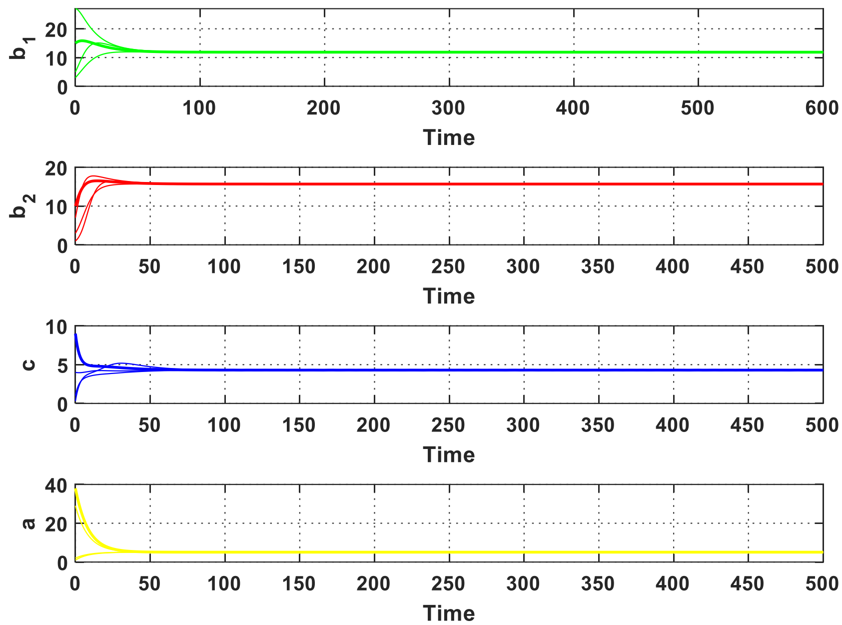

FIGURE 6. The behaviour of the system (1) with the data given by (5) and $\beta_0 = 0.2$.

Finally, the influence of decreasing antibiotic doses is specific in Figure 7. It is clear that the solution of system (1) converges asymptotically to the positive equilibrium point $s_{12} = (5.71, 14.43, 5.6, 4.23)$ for $\alpha = 0.5$. If this result is compared with case 4, it could be concluded that decreasing antibiotic doses positively impacts the growth rate of beneficial bacteria. As a result, this leads to decreased accumulated faces in the intestinal.

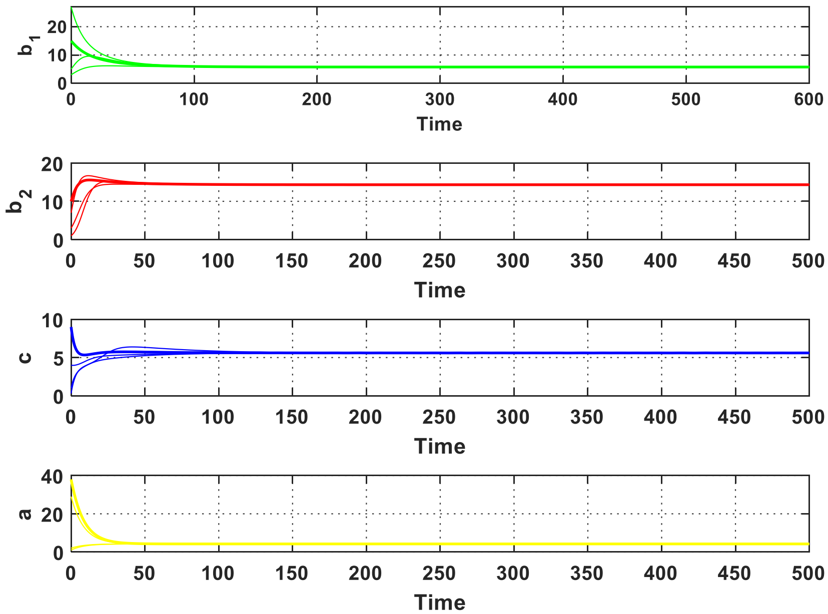

FIGURE 7. The behaviour of the system (1) with the data given by (4.5) and a $= 0.5$.

CONCLUSION

We draw up a mathematical model that depicts the effect of good and bad bacteria exposed to antibiotics and probiotics supplements in the large intestine. Based on the previous study, the model shows twelve non-negative equilibrium points. The local bifurcation at them has been studied using the Sotomayor theorem. The analysis result has shown that the transcritical and saddle-node bifurcation might occur at most of the equilibrium points. In contrast, the pitchfork bifurcation cannot be grown at any of them. In addition, the numerical section showed the good bacteria b_1 is more affected by antibiotics than harmful bacteria b_2. The reason is that antibiotics kill good and bad bacteria or prevent them from growing. Further, some good bacteria become harmful through mutations due to antibiotic exposure. As a result, the lack of good bacteria causes many problems, such as increased gases in the intestines, which causes discomfort and flatulence, indigestion, chronic diarrhoea or constipation, or colon diseases caused by a decrease in the types of anti-inflammatory intestinal bacteria.

Overall, the system with the decrease in the antibiotic dose and regular taking of Probiotic supplements has a positive impact on maintaining a balance between the symbiotic bacteria in the intestine ecosystem. As a result, adding Probiotics supplements as a treatment method has important effects on the system: beneficial bacteria compete with harmful bacteria in the lumen and reduce intestinal wall permeability. This role might help stabilize the ecosystems in the intestinal.

REFERENCES

1. Jing T-Z, Qi F-H, Wang Z-Y. Most dominant roles of insect gut bacteria: digestion, detoxification, or essential nutrient provision? Microbiome. 2020;8(1):1–20.
2. Zhang Y-J, Li S, Gan R-Y, Zhou T, Xu D-P, Li H-B. Impacts of gut bacteria on human health and diseases. Int J Mol Sci. 2015;16(4):7493–519.
3. Million M, Lagier J-C, Yahav D, Paul M. Gut bacterial microbiota and obesity. Clin Microbiol Infect. 2013;19(4):305–13.
4. Angelucci F, Cechova K, Amlerova J, Hort J. Antibiotics, gut microbiota, and Alzheimer's disease. J Neuroinflammation. 2019;16(1):1–10.
5. Rong CG. Hopf bifurcation analysis: a frequency domain approach, vol. 21. World Scientific; 1996.
6. Jawad S, Winter M, Rahman Z-ASA, Al-Yasir YIA, Zeb A. Dynamical Behavior of a Cancer Growth Model with Chemotherapy and Boosting of the Immune System. Mathematics. 2023;11(2):406.
7. Mukherjee D. Persistence and global stability of a population in a polluted environment with delay. J Biol Syst. 2002;10(03):225–32.
8. Ibargüen-Mondragón E, Mosquera S, Cerón M, Burbano-Rosero EM, Hidalgo-Bonilla SP, Esteva L, et al. Mathematical modeling on bacterial resistance to multiple antibiotics caused by spontaneous mutations. BioSystems. 2014;117:60–7.
9. Daşbaşı B, Öztürk İ. Mathematical modelling of bacterial resistance to multiple antibiotics and immune system response. Springerplus. 2016;5(1):1–17.
10. Crouch R, Haines* C. Mathematical modelling: Transitions between the real world and the mathematical model. Int J Math Educ Sci Technol. 2004;35(2):197–206.
11. Hanavan Jr EP. A mathematical model of the human body. Air Force Aerospace Medical Research Lab Wright-patterson AFB OH; 1964.
12. Walker AJ, Glen DM, Shewry PR. Bacteria associated with the digestive system of the slug Deroceras reticulatum are not required for protein digestion. Soil Biol Biochem. 1999;31(10):1387–94.
13. Hassan SK, Jawad SR. The Effect of Mutual Interaction and Harvesting on Food Chain Model. Iraqi J Sci. 2022;2641–9.
14. Ahmed M, Jawad S. The role of antibiotics and probiotics supplements on the stability of gut flora bacteria interactions. Commun Math Biol Neurosci. 2023;2023:Article-ID.
15. Fusco G, Iannelli M, Salvadori L. Advanced Topics in the Theory of Dynamical Systems: Notes and Reports in Mathematics in Science and Engineering, vol. 6. Vol. 6. Elsevier; 2016. Biol Neurosci. 2023;2023:Article-ID.