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STATISTICAL PROPERTIES OF GENERALIZED EXPONENTIAL RAYLEIGH DISTRIBUTION

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Abstract This paper demonstrates the construction of a modern generalized Exponential Rayleigh distribution by merging two distributions with a single parameter. The "New generalized Exponential-Rayleigh distribution" specifies joining the Reliability function of exponential pdf with the Reliability function of Rayleigh pdf, and then adding a shape parameter for this distribution. Finally, the mathematical and statistical characteristics of such a distribution are accomplished.

1 Introduction

Application of statistics plays an important role in our divergent phenomenon life, especially in engineering and medicine. Researchers depend on many procedures to find the new or mixture and compose distributions. In 2021, Iden H.H. and Lamyaa K.H. introduced a new class of exponential Rayleigh distribution defined and derived all properties of this distribution [6]. In 2021, Iden H.H. and Lamyaa K.H. applied the simulation technique for non-Bayesian estimated parameters and survival function for exponential Rayleigh distribution [2]. In 2021, Iden H.H. and Lamyaa K.H. applied maximum likelihood method to estimate values for survival function and Hazard function based on real data for lung cancer and stomach cancer obtained from Iraqis [3]. The exponential Rayleigh distribution contained two scale parameters. Now the aim of this paper is to compose the shape parameter to exponential Rayleigh distribution called ‘generalized exponential Rayleigh distribution’ which contain three parameters, one is shape and others are scale depending on the heavy tail distribution. The organization of this thesis include the mathematical structure of the composition distribution. It also define the probability density function, cumulative distribution, survival function, Hazard function and all statistical and mathematical property of this composition distribution.

Theorem 1.1. *Let y be a random variable with Exponential-Rayleigh distribution as a following*

$$f\left(y;\alpha,\beta\right)=\begin{cases}\left(\alpha+\beta y\right)e^{-\left(\alpha y+\frac{\beta}{2}y^{2}\right)} & y\geq 0\\ 0 & o.w\end{cases} \tag{1.1}$$

$$\Omega=\{(\alpha,\beta):\alpha>0,\beta>0\ \textit{where}\ \alpha,\beta\ \ \textit{are scale parameter}\}$$

If $x=y^{\lambda}$ then x has the generalized Exponential-Rayleigh distribution

$$f\left(y;\alpha,\beta\right)=\left(\alpha+\beta y\right)e^{-\left(\alpha y+\frac{\beta}{2}y^{2}\right)}\qquad y\geq 0$$

Proof.

$$x=y^{\lambda}\rightarrow y=x^{\frac{1}{\lambda}},\qquad \frac{dy}{dx}=\frac{1}{\lambda}x^{\frac{1}{\lambda}-1}$$

$$f\left(x;\alpha,\beta,\lambda\right)=f\left(y;\alpha,\beta\right)\frac{dy}{dx}$$

Where $|J|=|\frac{dy}{dx}|$ is Jacobian criterion

$$= \left(\frac{\alpha}{\lambda} x^{\frac{1}{\lambda}-1} + \frac{\beta}{\lambda} x^{\frac{2}{\lambda}-1} \right) e^{-\left(\alpha x^{\frac{1}{\lambda}} + \frac{\beta}{2} x^{\frac{2}{\lambda}} \right)} \qquad x \geq 0$$

$\Omega = \{(\alpha, \beta, \lambda) : \alpha, \beta, \lambda > 0$ where α, β are scale parameter and λ is a shape parameter$\}$

Then the function in 1.1 is probability density function for generalized Exponential-Rayleigh distribution. To prove this function is probability density function

$$\int_0^\infty f(x; \alpha, \beta, \lambda)\, dx = 1$$

$$\to \int_0^\infty e^{-\left(\alpha x^{\frac{1}{\lambda}} + \frac{\beta}{2} x^{\frac{2}{\lambda}} \right)} \left[- \left(\alpha \frac{1}{\lambda} x^{\frac{1}{\lambda}-1} + \frac{\beta}{2} \frac{2}{\lambda} x^{\frac{2}{\lambda}-1} \right) \right] dx = 1$$

$$\to - \left[e^{-\left(\alpha x^{\frac{1}{\lambda}} + \frac{\beta}{2} x^{\frac{2}{\lambda}} \right)} \right]_0^\infty = 1$$

□

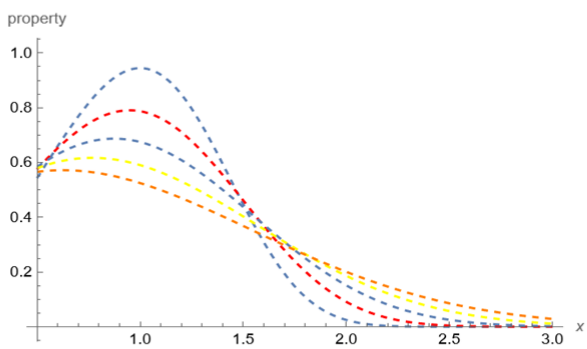

Figure 1. The density function of generalized Exponential-Rayleigh distribution with scale parameter $(\alpha = 0.5, \beta = 0.5)$ and different value of shape $(\lambda = 0.5, 0.6, 0.7, 0.8, 0.9)$.

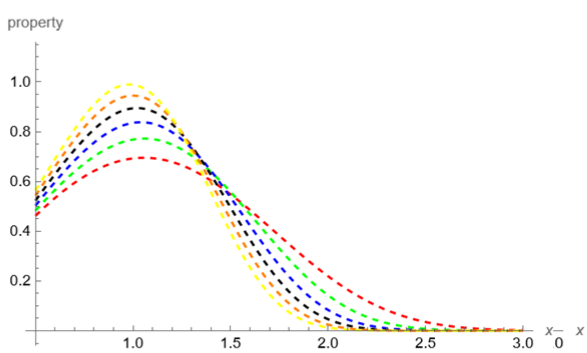

Figure 2. The density function of generalized Exponential-Rayleigh distribution with scale parameter $(\alpha = 0.5)$ shape, $(\lambda = 0.5)$ and different scale $(\beta = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6)$.

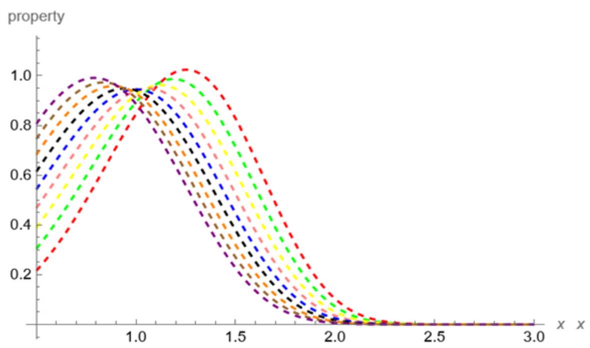

Figure 3. The density function of generalized Exponential-Rayleigh distribution with scale parameter $(\beta = 0.5)$ shape, $(\lambda = 0.5)$ and different scale $(\alpha = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9)$.

Corollary 1.2. *The cumulative distribution function for generalized Exponential- Rayleigh distribution is:*

$$F\left(x;\alpha,\beta,\lambda\right) = 1 - e^{-\left(\alpha x^{\frac{1}{\lambda}} + \frac{\beta}{2} x^{\frac{2}{\lambda}}\right)}$$

Proof.

$$F\left(x;\alpha,\beta,\lambda\right) = pr(X \leq x)$$

Where X random variable and$x$ is the value of random variable

$$F\left(x;\alpha,\beta,\lambda\right) = \int_0^x f\left(u;\alpha,\beta,\lambda\right) du = \left[e^{-\left(\alpha u^{\frac{1}{\lambda}} + \frac{\beta}{2} u^{\frac{2}{\lambda}}\right)}\right]_0^x, \text{ then}$$

$$F\left(x;\alpha,\beta,\lambda\right) = 1 - e^{-\left(\alpha x^{\frac{1}{\lambda}} + \frac{\beta}{2} x^{\frac{2}{\lambda}}\right)} \qquad x \geq 0$$

□

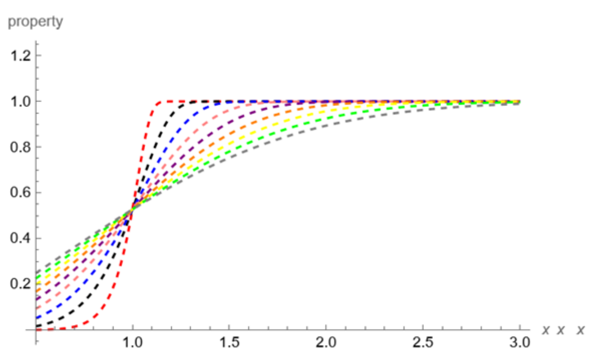

Figure 4. The cumulative function with constant scale parameter $(\alpha = 0.5, \beta = 0.5)$ and different shape $\lambda = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$ parameter value

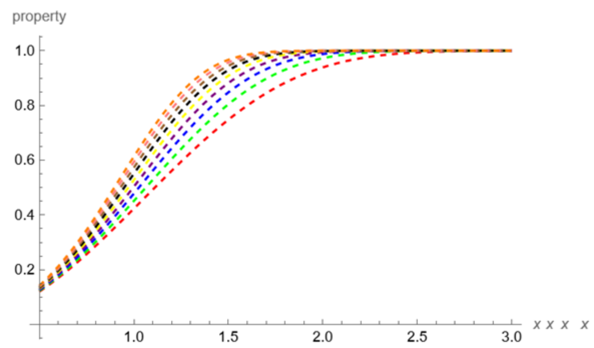

Figure 5. The cumulative function with constant scale parameter $(\alpha = 0.5)$, shape $(\lambda = 0.5)$ and different scale $(\beta = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9)$ parameter value

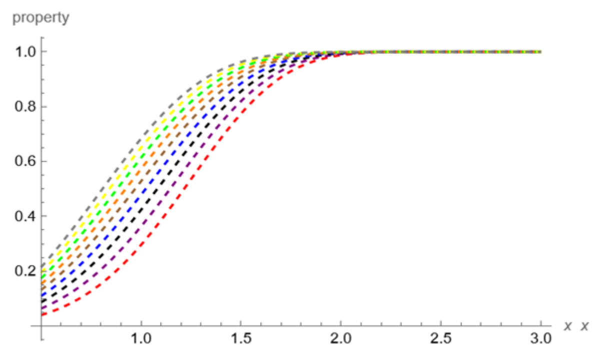

Figure 6. The cumulative function with constant scale parameter $(\beta = 0.5)$, shape $(\lambda = 0.5)$ and different scale $(\alpha = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9)$ parameter value

Then the reliability function of this distribution is;

$$R(x; \alpha, \beta, \lambda) = 1 - F(x; \alpha, \beta, \lambda) = 1 - 1 + e^{-\left(\alpha x^{\frac{1}{\lambda}} + \frac{\beta}{2} x^{\frac{2}{\lambda}}\right)} = e^{-\left(\alpha x^{\frac{1}{\lambda}} + \frac{\beta}{2} x^{\frac{2}{\lambda}}\right)} \quad \text{x} \geq 0$$

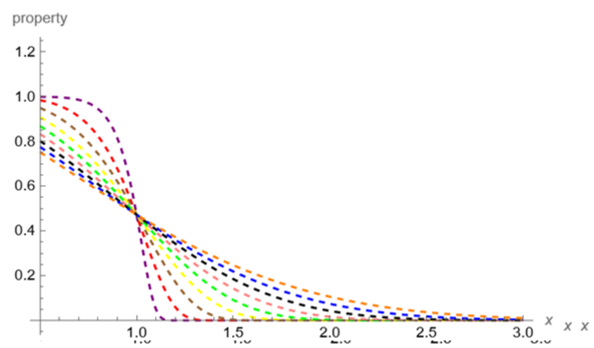

Figure 7. The Reliability function of generalized Exponential-Rayleigh distribution with scale parameter $(\alpha = 0.5, \ \beta = 0.5)$ and different value of shape $(\lambda = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9)$

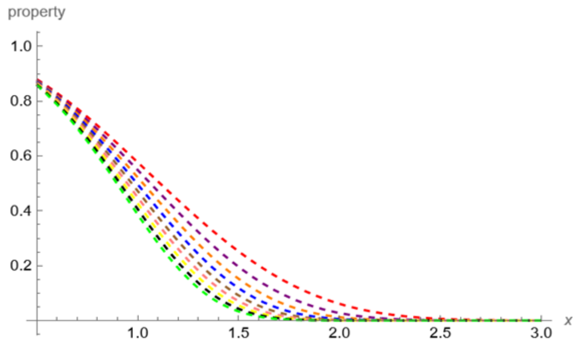

Figure 8. The Reliability function of generalized Exponential-Rayleigh distribution with scale parameter $(\alpha = 0.5)$, shape $(\lambda = 0.5)$ and different scale $(\beta = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9)$

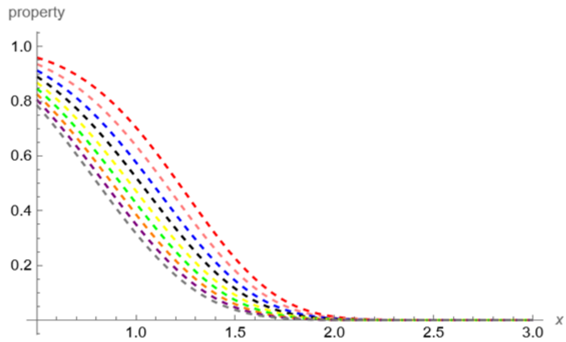

Figure 9. The Reliability function of generalized Exponential-Rayleigh distribution with scale parameter $(\beta = 0.5)$, shape $(\lambda = 0.5)$ and different scale $(\alpha = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9)$

Then the Hazard rate function for this distribution is:

$$h(x;\alpha,\beta,\lambda) = \frac{f(x;\alpha,\beta,\lambda)}{R(x;\alpha,\beta,\lambda)} = \frac{\left(\frac{\alpha}{\lambda}x^{\frac{1}{\lambda}-1} + \frac{\beta}{\lambda}x^{\frac{2}{\lambda}-1}\right)e^{-\left(\alpha x^{\frac{1}{\lambda}} + \frac{\beta}{2}x^{\frac{2}{\lambda}}\right)}}{e^{-\left(\alpha x^{\frac{1}{\lambda}} + \frac{\beta}{2}x^{\frac{2}{\lambda}}\right)}}$$

$$= \frac{\alpha}{\lambda}x^{\frac{1}{\lambda}-1} + \frac{\beta}{\lambda}x^{\frac{2}{\lambda}-1} \quad x \geq 0$$

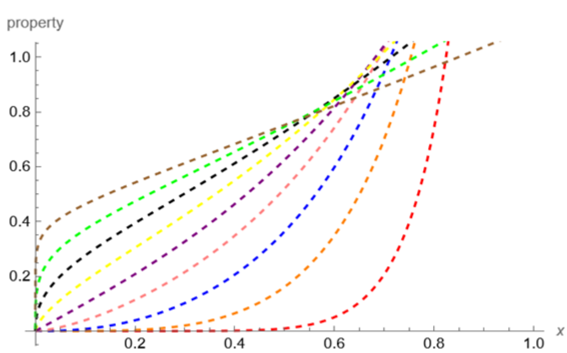

Figure 10. The Hazard rate function with constant scale $(\alpha = 0.5, \beta = 0.5)$ and different shape $\lambda = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$ parameter value.

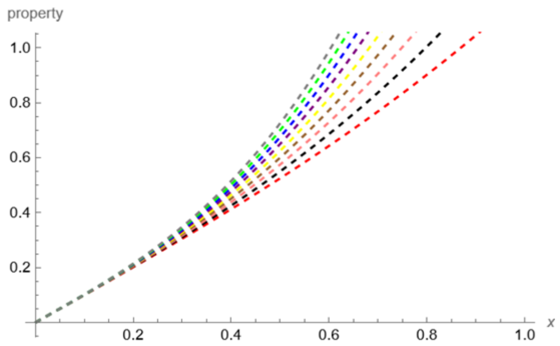

Figure 11. The Hazard rate function with constant scale $(\alpha = 0.5)$, shape $(\lambda = 0.5)$ and different scale $(\beta = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9)$ parameter value.

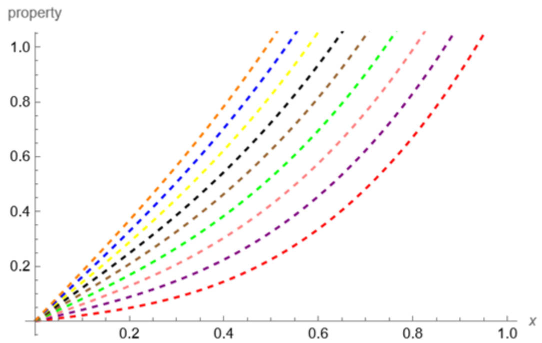

Figure 12. The Hazard rate function with constant scale $(\beta = 0.5)$, shape $(\lambda = 0.5)$ and different scale $(\alpha = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9)$

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Then the cumulative Hazard rate function for this distribution is:

$$H\left(x;\alpha,\beta,\lambda\right) = \frac{f\left(x;\alpha,\beta,\lambda\right)}{F\left(x;\alpha,\beta,\lambda\right)} = \frac{\left(\frac{\alpha}{\lambda}x^{\frac{1}{\lambda}-1} + \frac{\beta}{\lambda}x^{\frac{2}{\lambda}-1}\right)}{\left[e^{\left(\alpha x^{\frac{1}{\lambda}} + \frac{\beta}{2}x^{\frac{2}{\lambda}}\right)} - 1\right]} \qquad x \geq 0$$

Theorem 1.3. *The r th moment about the origin of generalized Exponential-Rayleigh distribution is:*

$$E(x^r) = \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!}\left[\frac{\alpha}{2}\left(\frac{2}{\beta}\right)^{\frac{n+\lambda r+1}{2}} \Gamma\left(\frac{n+\lambda r+1}{2}\right) + \left(\frac{2}{\beta}\right)^{\frac{n+\lambda r}{2}} \Gamma\left(\frac{n+\lambda r+2}{2}\right)\right]$$

Proof.

$$E(x^r) = \int_0^{\infty} x^r \;\; f(x;\; \alpha;\beta;\lambda)dx \quad , \; x > \; 0$$

$$= \int_0^{\infty} x^r \; \left(\frac{\alpha}{\lambda}x^{\frac{1}{\lambda}-1} + \frac{\beta}{\lambda}x^{\frac{2}{\lambda}-1}\right) e^{-\alpha x^{\frac{1}{\lambda}}} . \; e^{-\frac{\beta}{2}x^{\frac{2}{\lambda}}} \, \mathrm{dx}$$

By using Taylor series, then

$$e^{-\alpha x^{\frac{1}{\lambda}}} = \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} x^{\frac{n}{\lambda}}$$

$$= \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} \;\; \left[\int_0^{\infty} \frac{\alpha}{\lambda} x^{\frac{n+1}{\lambda}+r-1} e^{-\frac{\beta}{2}x^{\frac{2}{\lambda}}} dx + \int_0^{\infty} \frac{\beta}{\lambda} x^{\frac{2}{\lambda}-1} \; x^{\frac{n+2}{\lambda}+r-1} e^{-\frac{\beta}{2}x^{\frac{2}{\lambda}}} dx\right]$$

Let $K_1 = \int_0^\infty \frac{\alpha}{\lambda} x^{\frac{n+1}{\lambda}+r-1} e^{-\frac{\beta}{2}x^{\frac{2}{\lambda}}} dx$

Assume that $y = \frac{\beta}{2}x^{\frac{2}{\lambda}}$, $\left(\frac{2y}{\beta}\right)^{\frac{\lambda}{2}} = x$, then

$$\frac{dx}{dy} = \frac{\lambda}{2}\left(\frac{2}{\beta}\right)^{\frac{\lambda}{2}} y^{\frac{\lambda}{2}-1}$$

$$\to K_1 = \frac{\alpha}{\lambda}\int_0^\infty \left[\left(\frac{2y}{\beta}\right)^{\frac{\lambda}{2}}\right]^{\frac{n+1}{\lambda}+r-1} e^{-y}\frac{\lambda}{2}\left(\frac{2}{\beta}\right)^{\frac{\lambda}{2}} y^{\frac{\lambda}{2}-1}dy$$
$$= \frac{\alpha}{2}\left(\frac{2}{\beta}\right)^{\frac{n+1}{2}+\frac{\lambda\, r}{2}} \int_0^\infty (y)^{\frac{n+\lambda r-1}{2}} e^{-y}dy$$

The integral looks like gamma function, then $\to \alpha = \frac{n+\lambda r+1}{2}$, then

$$K_1 = \frac{\alpha}{2}\left(\frac{2}{\beta}\right)^{\frac{n+\lambda r+1}{2}} \Gamma\left(\frac{n+\lambda r+1}{2}\right).$$

Now, Assume that

$$K_2 = \int_0^\infty \frac{\beta}{\lambda} x^{\frac{n+2}{\lambda}+r-1} e^{-\frac{\beta}{2}x^{\frac{2}{\lambda}}} dx$$

Assume that $y = \frac{\beta}{2}x^{\frac{2}{\lambda}}$ $\to \left(\frac{2y}{\beta}\right)^{\frac{\lambda}{2}} = x$, then $\frac{dx}{dy} = \frac{\lambda}{2}\left(\frac{2}{\beta}\right)^{\frac{\lambda}{2}} y^{\frac{\lambda}{2}-1}$

$$K_2 = \frac{\beta}{2}\int_0^\infty \left(\frac{2y}{\beta}\right)^{\frac{n+2}{2}+\frac{\lambda\, r}{2}-\frac{\lambda}{2}} e^{-y}\left(\frac{2}{\beta}\right)^{\frac{\lambda}{2}} y^{\frac{\lambda}{2}-1}dy = \frac{\beta}{2}\left(\frac{2}{\beta}\right)^{\frac{n+2+\lambda r}{2}} \int_0^\infty (y)^{\frac{n+\lambda r}{2}} e^{-y}dy$$

The integral looks like gamma function, then $\to \alpha = \frac{n+\lambda r+2}{2}$. So,

$$K_2 = \left(\frac{2}{\beta}\right)^{\frac{n+\lambda n}{2}} \Gamma\left(\frac{n+\lambda n+2}{2}\right)$$
$$\to E(x^r) = \sum_{n=0}^{\infty}\frac{(-\alpha)^n}{n!}\left[\frac{\alpha}{2}\left(\frac{2}{\beta}\right)^{\frac{n+\lambda r+1}{2}}\Gamma\left(\frac{n+\lambda r+1}{2}\right) + \left(\frac{2}{\beta}\right)^{\frac{n+\lambda r}{2}}\Gamma\left(\frac{n+\lambda r+2}{2}\right)\right]$$

The Mean of this distribution is when $r = 1$

$$E(x) = \sum_{n=0}^{\infty}\frac{(-\alpha)^n}{n!}\left[\frac{\alpha}{2}\left(\frac{2}{\beta}\right)^{\frac{n+\lambda+1}{2}}\Gamma\left(\frac{n+\lambda+1}{2}\right) + \left(\frac{2}{\beta}\right)^{\frac{n+\lambda}{2}}\Gamma\left(\frac{n+\lambda+2}{2}\right)\right]$$
$$\to Var(x) = \sum_{n=0}^{\infty}\frac{(-\alpha)^n}{n!}\left(\frac{2}{\beta}\right)^{\frac{n+2\lambda}{2}}\left[\frac{\alpha}{\sqrt{2\beta}}\Gamma\left(\frac{n+2\lambda+1}{2}\right) + \Gamma\left(\frac{n+2\lambda+2}{2}\right)\right] -$$
$$\left[\sum_{n=0}^{\infty}\frac{(-\alpha)^n}{n!}\left(\frac{2}{\beta}\right)^{\frac{n+\lambda}{2}}\left[\frac{\alpha}{\sqrt{2\beta}}\Gamma\left(\frac{n+\lambda+1}{2}\right) + \Gamma\left(\frac{n+\lambda+2}{2}\right)\right]\right]^2$$

□

Theorem 1.4. *The moment generating function for generalized Exponential-Rayleigh distribution is:*

$$\mu_x(t) == \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{k^m}{m!}\frac{(-\alpha)^n}{n!}\left(\frac{2}{\beta}\right)^{\lambda m+n+\frac{3}{2}}\left(\frac{\alpha+\beta}{2}\right)\left[\Gamma\left(\frac{\lambda m+n+1}{2}\right) + \Gamma\left(\frac{\lambda m+n+2}{2}\right)\right]$$

Proof.

$$\mu_x(t) = \int_0^\infty e^{\text{tx}}\, \text{f}(x;\alpha,\beta,\lambda)\, dx$$

Recall that

$$e^{-\alpha x^{\frac{1}{\lambda}}} = \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} x^{\frac{n}{\lambda}}$$

Then

$$= \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} \int_0^\infty e^{\text{tx}} \left(\frac{\alpha}{\lambda} x^{\frac{1}{\lambda}-1} + \frac{\beta}{\lambda} x^{\frac{2}{\lambda}-1}\right) x^{\frac{n}{\lambda}}.e^{-\frac{\beta}{2} x^{\frac{2}{\lambda}}}\, dx$$

Recall that

$$e^{\text{tx}} = \sum_{m=0}^{\infty} \frac{k^m}{m!} x^m$$

Then

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{k^m}{m!}\frac{(-\alpha)^n}{n!} \left[\int_0^\infty x^{m+\frac{n}{\lambda}} \frac{\alpha}{\lambda} x^{\frac{1}{\lambda}-1}\, e^{-\frac{\beta}{2} x^{\frac{2}{\lambda}}}\, dx + \int_0^\infty x^{m+\frac{n}{\lambda}} \frac{\beta}{\lambda} x^{\frac{2}{\lambda}-1} e^{-\frac{\beta}{2} x^{\frac{2}{\lambda}}}\, dx\right]$$

Assume that

$$L_1 = \int_0^\infty x^{m+\frac{n}{\lambda}} \frac{\alpha}{\lambda} x^{\frac{1}{\lambda}-1}\, e^{-\frac{\beta}{2} x^{\frac{2}{\lambda}}}\, dx$$

Let y=$\frac{\beta}{2}$, $\quad (\frac{2y}{\beta})^{\frac{\lambda}{2}} = x,$ then

$$\frac{dx}{dy} = \left(\frac{2}{\beta}\right)^{\frac{\lambda}{2}} \frac{\lambda}{2} y^{\frac{\lambda}{2}-1}, \qquad L_1 = \frac{\alpha}{2}\left(\frac{2}{\beta}\right)^{\frac{\lambda m+n+1}{2}} \int_0^\infty (y)^{\frac{\lambda\, m}{2}+\frac{n}{2}+\frac{1}{2}-1} e^{-y} dy$$

The integral looks like gamma function

$$\rightarrow \alpha = \frac{\lambda m+n+1}{2}$$

Then

$$L_1 = \frac{\alpha}{2}\left(\frac{2}{\beta}\right)^{\frac{\lambda m+n+1}{2}} \Gamma\left(\frac{\lambda m+n+1}{2}\right)$$

Now, we drive the second integral, then

$$L_2 = \int_0^\infty x^{m+\frac{n}{\lambda}} \frac{\beta}{\lambda} x^{\frac{2}{\lambda}-1} e^{-\frac{\beta}{2} x^{\frac{2}{\lambda}}}\, dx$$

Let y=$\frac{\beta}{2} x^{\frac{2}{\lambda}}$, $\quad (\frac{2y}{\beta})^{\frac{\lambda}{2}} = x,$ then
$\frac{dx}{dy} = \left(\frac{2}{\beta}\right)^{\frac{\lambda}{2}} \frac{\lambda}{2} y^{\frac{\lambda}{2}-1}$, $L_2 = \frac{\beta}{2}\left(\frac{2}{\beta}\right)^{\frac{\lambda m+n+1}{2}} \int_0^\infty (y)^{\frac{\lambda\, m}{2}+\frac{n}{2}} e^{-y} dy$
The integral looks like gamma function

$$\rightarrow \alpha = \frac{\lambda\, m}{2} + \frac{n}{2} + 1$$

Then

$$L_2 = \frac{\beta}{2}\left(\frac{2}{\beta}\right)^{\frac{\lambda m+n+2}{2}} \Gamma\left(\frac{\lambda\, m}{2} + \frac{n}{2} + 1\right)$$

$$\mu_x(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{k^m}{m!}\frac{(-\alpha)^n}{n!}\left(\frac{2}{\beta}\right)^{\lambda m+n+\frac{3}{2}} \left(\frac{\alpha+\beta}{2}\right)\left[\Gamma\left(\frac{\lambda m+n+1}{2}\right) + \Gamma\left(\frac{\lambda m+n+2}{2}\right)\right]$$

□

Corollary 1.5. *The mode of the generalized Exponential-Rayleigh distribution is:*

$$S_1 = \frac{\alpha}{3\beta} + \frac{2^{1/3}\left(2\alpha^2\beta^2 - 6\beta^3 + 3\beta^3\lambda\right)}{3\beta^2\left(7\alpha^3\beta^3 - 45\alpha\beta^4 + 36\alpha\beta^4\lambda + \sqrt{4\left(2\alpha^2\beta^2 - 6\beta^3 + 3\beta^3\lambda\right)^3 + \left(7\alpha^3\beta^3 - 45\alpha\beta^4 + 36\alpha\beta^4\lambda\right)^2}\right)^{1/3}} - \frac{\left(7\alpha^3\beta^3 - 45\alpha\beta^4 + 36\alpha\beta^4\lambda + \sqrt{4\left(2\alpha^2\beta^2 - 6\beta^3 + 3\beta^3\lambda\right)^3 + \left(7\alpha^3\beta^3 - 45\alpha\beta^4 + 36\alpha\beta^4\lambda\right)^2}\right)^{1/3}}{32^{1/3}\beta^2}$$

$$S_2 = \frac{\alpha}{3\beta} - \frac{\left(1 + i\sqrt{3}\right)\left(2\alpha^2\beta^2 - 6\beta^3 + 3\beta^3\lambda\right)}{32^{\frac{2}{3}}\beta^2\left(7\alpha^3\beta^3 - 45\alpha\beta^4 + 36\alpha\beta^4\lambda + \sqrt{4\left(2\alpha^2\beta^2 - 6\beta^3 + 3\beta^3\lambda\right)^3 + \left(7\alpha^3\beta^3 - 45\alpha\beta^4 + 36\alpha\beta^4\lambda\right)^2}\right)^{\frac{1}{3}}} + \frac{\left(1 - i\sqrt{3}\right)\left(7\alpha^3\beta^3 - 45\alpha\beta^4 + 36\alpha\beta^4\lambda + \sqrt{4\left(2\alpha^2\beta^2 - 6\beta^3 + 3\beta^3\lambda\right)^3 + \left(7\alpha^3\beta^3 - 45\alpha\beta^4 + 36\alpha\beta^4\lambda\right)^2}\right)^{1/3}}{62^{1/3}\beta^2}$$

$$S_3 = \frac{\alpha}{3\beta} - \frac{\left(1 - i\sqrt{3}\right)\left(2\alpha^2\beta^2 - 6\beta^3 + 3\beta^3\lambda\right)}{32^{2/3}\beta^2\left(7\alpha^3\beta^3 - 45\alpha\beta^4 + 36\alpha\beta^4\lambda + \sqrt{4\left(2\alpha^2\beta^2 - 6\beta^3 + 3\beta^3\lambda\right)^3 + \left(7\alpha^3\beta^3 - 45\alpha\beta^4 + 36\alpha\beta^4\lambda\right)^2}\right)^{\frac{1}{3}}} + \frac{\left(1 + i\sqrt{3}\right)\left(7\alpha^3\beta^3 - 45\alpha\beta^4 + 36\alpha\beta^4\lambda + \sqrt{4\left(2\alpha^2\beta^2 - 6\beta^3 + 3\beta^3\lambda\right)^3 + \left(7\alpha^3\beta^3 - 45\alpha\beta^4 + 36\alpha\beta^4\lambda\right)^2}\right)^{1/3}}{62^{1/3}\beta^2}$$

Proof.

$$\acute{f}\left(t;\alpha,\beta,\lambda\right) = \frac{\partial f\left(t;\alpha,\beta,\lambda\right)}{\partial t} = 0$$

$$\frac{1}{\lambda}t^{\frac{1}{\lambda}-2}\left[\frac{\alpha}{\lambda} - \alpha + \frac{2\beta}{\lambda}t^{\frac{1}{\lambda}} - \beta t^{\frac{1}{\lambda}} - \frac{\alpha^2}{\lambda}t^{\frac{1}{\lambda}} - \frac{2\alpha\beta}{\lambda}t^{\frac{2}{\lambda}} - \frac{\beta^2}{\lambda}t^{\frac{3}{\lambda}}\right] = 0$$

Let $t \neq 0$ then $\frac{1}{\lambda}t^{\frac{1}{\lambda}-2} \neq 0$
If $\frac{1}{\lambda}t^{\frac{1}{\lambda}-2} = a$, a constant

$$t = (a\lambda)^{\frac{\lambda}{1-2\lambda}}$$

$$\left(\frac{\alpha - \alpha\lambda}{\lambda}\right) + \left(\frac{2\beta}{\lambda} - \beta - \frac{\alpha^2}{\lambda}\right)t^{\frac{1}{\lambda}} - \frac{2\alpha\beta}{\lambda}t^{\frac{2}{\lambda}} - \frac{\beta^2}{\lambda}t^{\frac{3}{\lambda}} = 0$$

Let $t^{\frac{1}{\lambda}} = s \rightarrow t^{\frac{2}{\lambda}} = s^2, t^{\frac{3}{\lambda}} = s^3$

$$\left(\frac{\alpha - \alpha\lambda}{\lambda}\right) + \left(\frac{2\beta}{\lambda} - \beta - \frac{\alpha^2}{\lambda}\right)s - \frac{2\alpha\beta}{\lambda}s^2 - \frac{\beta^2}{\lambda}s^3 = 0$$

Maximize the distribution

$$S_1 = \frac{\alpha}{3\beta} + \frac{2^{1/3}\left(2\alpha^2\beta^2 - 6\beta^3 + 3\beta^3\lambda\right)}{3\beta^2\left(7\alpha^3\beta^3 - 45\alpha\beta^4 + 36\alpha\beta^4\lambda + \sqrt{4\left(2\alpha^2\beta^2 - 6\beta^3 + 3\beta^3\lambda\right)^3 + \left(7\alpha^3\beta^3 - 45\alpha\beta^4 + 36\alpha\beta^4\lambda\right)^2}\right)^{1/3}} - \frac{\left(7\alpha^3\beta^3 - 45\alpha\beta^4 + 36\alpha\beta^4\lambda + \sqrt{4\left(2\alpha^2\beta^2 - 6\beta^3 + 3\beta^3\lambda\right)^3 + \left(7\alpha^3\beta^3 - 45\alpha\beta^4 + 36\alpha\beta^4\lambda\right)^2}\right)^{1/3}}{32^{1/3}\beta^2}$$

$$S_2 = \frac{\alpha}{3\beta} - \frac{\left(1 + i\sqrt{3}\right)\left(2\alpha^2\beta^2 - 6\beta^3 + 3\beta^3\lambda\right)}{32^{\frac{2}{3}}\beta^2\left(7\alpha^3\beta^3 - 45\alpha\beta^4 + 36\alpha\beta^4\lambda + \sqrt{4\left(2\alpha^2\beta^2 - 6\beta^3 + 3\beta^3\lambda\right)^3 + \left(7\alpha^3\beta^3 - 45\alpha\beta^4 + 36\alpha\beta^4\lambda\right)^2}\right)^{\frac{1}{3}}} + \frac{\left(1 - i\sqrt{3}\right)\left(7\alpha^3\beta^3 - 45\alpha\beta^4 + 36\alpha\beta^4\lambda + \sqrt{4\left(2\alpha^2\beta^2 - 6\beta^3 + 3\beta^3\lambda\right)^3 + \left(7\alpha^3\beta^3 - 45\alpha\beta^4 + 36\alpha\beta^4\lambda\right)^2}\right)^{1/3}}{62^{1/3}\beta^2}$$

$$S_3 = \frac{\alpha}{3\beta} - \frac{\left(1 - i\sqrt{3}\right)\left(2\alpha^2\beta^2 - 6\beta^3 + 3\beta^3\lambda\right)}{32^{2/3}\beta^2\left(7\alpha^3\beta^3 - 45\alpha\beta^4 + 36\alpha\beta^4\lambda + \sqrt{4\left(2\alpha^2\beta^2 - 6\beta^3 + 3\beta^3\lambda\right)^3 + \left(7\alpha^3\beta^3 - 45\alpha\beta^4 + 36\alpha\beta^4\lambda\right)^2}\right)^{\frac{1}{3}}} + \frac{\left(1 + i\sqrt{3}\right)\left(7\alpha^3\beta^3 - 45\alpha\beta^4 + 36\alpha\beta^4\lambda + \sqrt{4\left(2\alpha^2\beta^2 - 6\beta^3 + 3\beta^3\lambda\right)^3 + \left(7\alpha^3\beta^3 - 45\alpha\beta^4 + 36\alpha\beta^4\lambda\right)^2}\right)^{1/3}}{62^{1/3}\beta^2}$$

□

Corollary 1.6. *The median for the generalized Exponential-Rayleigh distribution the median.*

$$F(t) = \frac{1}{2} \to F(t; \alpha, \beta, \lambda) = \frac{1}{2}$$

$$1 - e^{-\left(\alpha t^{\frac{1}{\lambda}} + \frac{\beta}{2} t^{\frac{2}{\lambda}}\right)} = \frac{1}{2}$$

$$\to \alpha t^{\frac{1}{\lambda}} + \frac{\beta}{2} t^{\frac{2}{\lambda}} = \ln 2$$

Let $t^{\frac{1}{\lambda}} = x$ *and* $t^{\frac{2}{\lambda}} = x^2$

$$\to \alpha x + \frac{\beta}{2} x^2 - \ln 2 = 0$$

$$x = \frac{-(2\alpha) \mp \sqrt{4\alpha^2 + 8\beta \ln 2}}{2\beta}$$

$$\to t = \left(\frac{-2\alpha \mp \sqrt{4\alpha^2 + 5.544\beta}}{2\beta}\right)^{\lambda}$$

Corollary 1.7. *The Quantile function is:*

$$x = Q(u) = F^{-1}(v)$$

$$\rightarrow 2\alpha x^{\frac{1}{\beta}} + \lambda x^{\frac{2}{\beta}} - 2\ln(1-v) = 0$$

$$\rightarrow x = \left[\frac{-(2\alpha) \mp \sqrt{4\alpha^2 - 8\lambda\ln(1-v)}}{2\lambda}\right]^{\beta}$$

Corollary 1.8. *The Skewness and Kurtosis is:*

$$C.S = \frac{\sum_{n=0}^{\infty}\frac{(-\alpha)^n}{n!}\left[\frac{\alpha}{2}\left(\frac{2}{\beta}\right)^{\frac{n+3\lambda+1}{2}}\Gamma\left(\frac{n+3\lambda+1}{2}\right) + \left(\frac{2}{\beta}\right)^{\frac{n+3\lambda}{2}}\Gamma\left(\frac{n+3\lambda+2}{2}\right)\right]}{\sum_{n=0}^{\infty}\frac{(-\alpha)^n}{n!}\left[\frac{\alpha}{2}\left(\frac{2}{\beta}\right)^{\frac{n+3\lambda+1}{2}}\Gamma\left(\frac{n+3\lambda+1}{2}\right) + \left(\frac{2}{\beta}\right)^{\frac{n+3\lambda}{2}}\Gamma\left(\frac{n+2\lambda+2}{2}\right)\right]^{\frac{3}{2}}}$$

$$C.K = \frac{\sum_{n=0}^{\infty}\frac{(-\alpha)^n}{n!}\left[\frac{\alpha}{2}\left(\frac{2}{\beta}\right)^{\frac{n+4\lambda+1}{2}}\Gamma\left(\frac{n+4\lambda+1}{2}\right) + \left(\frac{2}{\beta}\right)^{\frac{n+4\lambda}{2}}\Gamma\left(\frac{n+4\lambda+2}{2}\right)\right]}{\left(\sum_{n=0}^{\infty}\frac{(-\alpha)^n}{n!}\left[\frac{\alpha}{2}\left(\frac{2}{\beta}\right)^{\frac{n+2\lambda+1}{2}}\Gamma\left(\frac{n+2\lambda+1}{2}\right) + \left(\frac{2}{\beta}\right)^{\frac{n+2\lambda}{2}}\Gamma\left(\frac{n+2\lambda+2}{2}\right)\right]\right)^2} - 3$$

Theorem 1.9. *Factorial moment generating function is:*

$$\mu_x(t) = E(t^x) = \sum \frac{(\ln t)^n}{n!}E(x^n)$$

Proof.

$$\mu_x(t) = E(t^x) = \int_0^{\infty} t^x\ f(x;\alpha,\beta,\lambda)\,dx$$

Recall that

$$t^x = e^{\ln t^x} = e^{x\ln t}$$

$$e^{x\ln t} = \sum_{r=0}^{\infty}\frac{(\ln t)^r}{r!}x^r$$

$$\mu_x(t) = E(t^x) = \sum_{r=0}^{\infty}\frac{(\ln t)^r}{r!}\ E(x^r)$$

By using Taylor series, then

$$e^{-\alpha x^{\frac{1}{\lambda}}} = \sum_{n=0}^{\infty}\frac{(-\alpha)^n}{n!}x^{\frac{n}{\lambda}}$$

$$E(x^n) = \sum_{n=0}^{\infty}\frac{(-\alpha)^n}{n!}\left[\int_0^{\infty}\frac{\alpha}{\lambda}\ x^{\frac{n+1}{\lambda}+n-1}\ .\ e^{-\frac{\beta}{2}x^{\frac{2}{\lambda}}}\,dx + \int_0^{\infty}\frac{\beta}{\lambda}\ x^{\frac{n+2}{\lambda}+n-1}\ .\ e^{-\frac{\beta}{2}x^{\frac{2}{\lambda}}}\,dx\right]$$

$$K_1 = \int_0^{\infty}\frac{\alpha}{\lambda}\ x^{\frac{n+1}{\lambda}+n-1}\ .\ e^{-\frac{\beta}{2}x^{\frac{2}{\lambda}}}\,dx$$

$$\rightarrow x = \left(\frac{2y}{\beta}\right)^{\frac{\lambda}{2}} \rightarrow x = \left(\frac{2}{\beta}\right)^{\frac{\lambda}{2}}(y)^{\frac{\lambda}{2}}$$

$$\rightarrow \frac{dx}{dy} = \left(\frac{2}{\beta}\right)^{\frac{\lambda}{2}}\frac{\lambda}{2}(y)^{\frac{\lambda}{2}-1}$$

Then

$$K_1 = \frac{\alpha}{2}\left(\frac{2}{\beta}\right)^{\frac{n+1}{2}+\frac{\lambda\, n}{2}} \int_0^{\infty} y^{\frac{n+\lambda n-1}{2}} \; . \, e^{-y} dy$$

The integral look like gamma function, then

$$\alpha = \frac{n+\lambda n+1}{2}$$

$$K_1 = \frac{\alpha}{2}\left(\frac{2}{\beta}\right)^{\frac{n+\lambda n+1}{2}} \int_0^{\infty} y^{\frac{n+\lambda n+1}{2}-1} \; . \, e^{-y} dy = \frac{\alpha}{2}\left(\frac{2}{\beta}\right)^{\frac{n+\lambda n+1}{2}} \Gamma\left(\frac{n+\lambda n+1}{2}\right)$$

$$K_2 = \int_0^{\infty} \frac{\beta}{\lambda}\, x^{\frac{n+2}{\lambda}+n-1} \; . \, e^{-\frac{\beta}{2} x^{\frac{2}{\lambda}}} dx$$

Let $y = \frac{\beta}{2} x^{\frac{2}{\lambda}} \rightarrow x = \left(\frac{2}{\beta}\right)^{\frac{\lambda}{2}} (y)^{\frac{\lambda}{2}}$

$$\rightarrow \frac{dx}{dy} = \left(\frac{2}{\beta}\right)^{\frac{\lambda}{2}} \frac{\lambda}{2} (y)^{\frac{\lambda}{2}-1}$$

$$\rightarrow \alpha = \frac{n+n\lambda+2}{2} = \left(\frac{2}{\beta}\right)^{\frac{n+n\lambda}{2}} \Gamma\left(\frac{n+n\lambda+2}{2}\right)$$

$$\mu_x(t) = E\left(t^x\right) = \sum \frac{(\ln t)^n}{n!} E\left(x^n\right)$$

□

Corollary 1.10. *The rth central moment about mean*

$$E(x-\mu)^r = \int_0^{\infty} (x-\mu)^r f(x)\, dx$$

Recall that

$$(a+b)^2 = \sum_{j=0}^{r} C_j^r \, a^j \, b^{n-j}$$

$$(x-\mu)^r = (x+(-\mu))^r = \sum_{r=0}^{n} C_r^n \, x^r \, (-\mu)^{n-r}$$

$$\rightarrow E(x-\mu)^r = \int_0^{\infty} \sum_{r=0}^{n} C_r^n \, x^r \, (-\mu)^{n-r} f(x)\, dx$$

1.1 Characteristic function

$$\varnothing_x(it) = E\left(e^{itx}\right) = \int_0^{\infty} e^{itx} f(x;\alpha,\beta,\lambda)\, dx\,, \qquad e^{itx} = \sum_{n=0}^{\infty} \frac{(it)^r}{n!} x^r$$

$$\rightarrow \varnothing_x(it) = E\left(e^{itx}\right) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} E\left(x^r\right).$$

1.2 Order statistic distribution

$$g(y_i) = \frac{n!}{(i-1)!\,(n-i)!}\,[F(y_i)]^{i-1}\,[1-F(y_i)]^{n-i}\,f(y_i)$$
$$= \frac{n!}{(i-1)!\,(n-i)!}\left[1-e^{-\left(\alpha y_i^{\frac{1}{\lambda}}+\frac{\beta}{2}y_i^{\frac{2}{\lambda}}\right)}\right]^{i-1}\left[e^{-\left(\alpha y_i^{\frac{1}{\lambda}}+\frac{\beta}{2}y_i^{\frac{2}{\lambda}}\right)}\right]^{n-i}\star$$
$$\left(\frac{\alpha}{\lambda}y_i^{\frac{1}{\lambda}-1}+\frac{\beta}{\lambda}y_i^{\frac{2}{\lambda}-1}\right)e^{-\left(\alpha y_i^{\frac{1}{\lambda}}+\frac{\beta}{2}y_i^{\frac{2}{\lambda}}\right)}$$

1.3 Cumulative distribution function for order statistic

$$MTTE = \int_0^\infty R(t)\,dx = \int_0^\infty e^{-\left(\alpha x^{\frac{1}{\lambda}}+\frac{\beta}{2}x^{\frac{2}{\lambda}}\right)}dx, \qquad e^{-\alpha x^{\frac{1}{\lambda}}} = \sum_{n=0}^{\infty}\frac{(-\alpha)^n}{n!}x^{\frac{n}{\lambda}}$$
$$\to MTTE = e^{-\alpha}\left(\frac{\lambda}{2}\right)\left(\frac{2}{\beta}\right)^{\alpha}\Gamma\left(\frac{n+\lambda}{2}\right).$$

2 Conclusion

The study proposed a compose exponential-Rayleigh distribution which called as "Generalized Rayleigh distribution " which is expand of Exponential-Rayleigh distribution in the analysis of data with real support.

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