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To cite this article: N A Hussein and L N M Tawfiq 2020 *J. Phys.: Conf. Ser.* **1530** 012098

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New Approach for Solving (1+1)-Dimensional Differential Equation

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Abstract. In this paper, the new approach to solve important type of partial differential equations based on (1+1) dimension was proposed. Analytical solitary wave combine with Adomian Decomposition method is suggested to get exact solution of nonlinear (1+1) dimension PDEs. Yields solution in rapid convergent series from easily computable terms to get exact solution, and in some cases, yields in few iteration we get exact solution. Moreover, this approach does not require any discretization or perturbations and therefore reduces the computations. The suggested procedure easy implementation yet highly accurate and rapidly converge to exact solution compares with the other methods. The methodology presented here is useful for strongly nonlinear problems.

1. Introduction

In the last three decades, the Adomian Decomposition Method (ADM) has confirmed successful in getting analytical solution of non-linear differential equations by obtained solution in terms of convergent power series [1]. This method does not require discretization of the variables or domain [2, 3]. The theoretical analysis of convergence for the series solution of ADM has been studied by Adomian [4], Cherrault [5], Cherrault et al. [6], and Chrysos et al. [7].

In recent years, there has been development in the application of ADM in solving partial differential equations (PDEs) with variable coefficients. For example, Wazwaz and Gorguis [8] also used the ADM solved linear (1+1) differential equations with variable coefficients. Soufyane and Boulmaf [9] applied ADM to get analytical solutions to the non-linear parabolic equation with variable physical parameters in time and space. Achouri and Omrani [10], applied the ADM to get numerical solutions for the damped generalized regularized long-wave equation (DGRLW) with variable coefficients, and Tawfiq et al. (11-14) used this method for solving different model equations.

It is observed however that there are only a limited number of studies being researched on the use of ADM for solving nonlinear wave equations. The aim of this paper is to get the analytical solution of this type of equations using new modification of ADM.

2. Combine ADM with Solitary Wave for Solving Nonlinear (1+1)-D Equation

In this section we describe the procedure of suggested new approach for solving nonlinear (1+1)-D equations with initial condition.

For a given NLPDEs with independent variables $X = (x, y, z, \dots, t)$ and dependent variable u :

$$G(u, u_x, u_y, u_t, u_{xy}, \dots) = 0 \quad \dots(1)$$

First, suppose the following new wave variable:

$$\xi = x - ct. \quad \dots(2)$$



Then the solution of this equation is of the form

$$u(x, t) = u(\xi) = f(x-t) \quad \dots(3)$$

Now, substitute equation (2) into equation (1) to obtain new form of equation since the equation transform from PDE to ODE. Hence we solve final form by ADM as follow:

1. Express the PDE, linear or nonlinear, in operator form.
2. Take the inverse operator to both sides of the Eq. (3).
3. Write $u(x, y)$ into a decomposition infinite series of the form:

$$u(X, t) = \sum_{n=0}^{\infty} u_n(X, t) \quad \dots (4)$$

4. $N_1 u$ = for $\sum_{n=0}^{\infty} A_{1n}$ and $N_2 u$ = the $\sum_{n=0}^{\infty} A_{2n}$ nonlinear term $\dots (5)$

Where A_n is Adomian polynomials can be computed by

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[F \sum_{i=0}^{\infty} (\lambda^i u_i) \right]_{\lambda=0}, \quad n = 0, 1, 2, \dots \quad (6)$$

3. Illustrative Examples

In this section we gave three examples to illustrate suggested modification, and implemented to the nonlinear wave equations:

Example 1

Conceder the following nonlinear 3rd order (1+1)-D dispersive equation formulated by Kdv as:

$$u_t - 6uu_t + u_{xxx} = 0 \quad (7)$$

such that $u(x, t)$ is can able derivative and $u \rightarrow 0$ as $|x| \rightarrow 0$

$$U_t - 6UU_x + U_{xxx} = 0$$

Let $U(x, t) = f(x - ct)$

$$U_t = -cf'$$

$$U_x = f', \quad U_{xx} = f'', \quad U_{xxx} = f'''$$

$$-cf' - ff' + f''' = 0, \quad Z = x - ct$$

$$-cf - 3f^2 + f'' = 0$$

$$-2cff' - 6f^2f' + 2f'f'' = 0$$

$$-cf^2 - 2f^3 + (f')^2 = 0$$

$$(f)'^2 = cf^2 + 2f^3 \tag{8}$$

$$f' = \pm\sqrt{cf^2 + 2f^3}$$

$$f' = \pm\sqrt{c} f \sqrt{1 + \frac{2f}{c}}$$

let $Y^2 = \frac{2f}{c}$, $f = \frac{c}{2}Y^2$, $f' = cYY'$

$$cYY' = \pm\sqrt{c} \frac{c}{2}Y^2\sqrt{1+Y^2}$$

$$\frac{Y'}{Y\sqrt{1+Y^2}} = \pm \frac{\sqrt{c}}{2}$$

$$\frac{dy}{y\sqrt{1+Y^2}} = \pm \frac{c}{2} dz$$

$$csch^{-1} Y = \pm \frac{\sqrt{c}}{2} z$$

$$Y = \pm csch \frac{\sqrt{c}}{2} z$$

$$\frac{2f}{c} = csch^2 \frac{c}{2} z$$

$f = \frac{c}{2} csch^2 \frac{c}{2} z$, the solution of equation (7) is

$$U = \frac{c}{2} csch^2 \frac{\sqrt{c}}{2} (x - ct)$$

This solution satisfy the equation (7), that is

$$f = \frac{c}{2} csch^2 \frac{c}{2} (z)$$

$$f' = \frac{c}{2} 2csch \frac{\sqrt{c}}{2} (z) (-csch \frac{c}{2} z \coth \frac{\sqrt{c}}{2} z) (\frac{\sqrt{c}}{2})$$

$$f' = \frac{-c\sqrt{c}}{2} csch^2 \frac{\sqrt{c}}{2} z \coth \frac{\sqrt{c}}{2} z, \text{ substitute } f, f' \text{ in eq. (8)}$$

$$-c \left[\frac{c^2}{4} csch^4 \frac{c}{2} z \right] - 2 \left[\frac{c^3}{8} csch^6 \frac{\sqrt{c}}{2} z \right] + \frac{c^3}{4} csch^4 \frac{\sqrt{c}}{2} z \coth^2 \frac{c}{2} z$$

$$\frac{-c^3}{4} csch^4 \frac{\sqrt{c}}{2} z - \frac{c^3}{4} csch^6 \frac{\sqrt{c}}{2} z + \frac{c^3}{4} csch^4 \frac{\sqrt{c}}{2} z (1 + csch^2 \frac{\sqrt{c}}{2} z)$$

$$\frac{-c^3}{4} csch^4 \frac{\sqrt{c}}{2} z - \frac{c^3}{4} csch^6 \frac{\sqrt{c}}{2} z + \frac{c^3}{4} csch^4 \frac{\sqrt{c}}{2} z + \frac{c^3}{4} csch^6 \frac{c}{2} z = 0$$

Thus $U = \frac{c}{2} csch^2 \frac{\sqrt{c}}{2} (x - ct)$ is representing the exact solution.

Also can be obtain the same solution of equation (7) by solving equation (9) as follows

$$-cf - 3f^2 + f'' = 0$$

$$\text{Let } p = f', \rightarrow p \frac{dp}{df} = f''$$

$$p \frac{dp}{df} = cf + 3f^2$$

$$\frac{p^2}{2} = \frac{c}{2}f^2 + f^3$$

$$p = \pm \sqrt{cf^2 + 2f^3}$$

$$f' = \pm \sqrt{cf^2 + 2f^3}$$

$$f' = \pm \sqrt{c} f \sqrt{1 + \frac{2f}{c}} \quad (9)$$

$$\text{let } Y^2 = \frac{2f}{c}$$

$$f = \frac{c}{2}Y^2$$

$$f' = cYY'$$

$$cYY' = \pm \sqrt{c} \frac{c}{2} Y^2 \sqrt{1 + Y^2}$$

$$\frac{Y'}{Y\sqrt{1+Y^2}} = \pm \frac{\sqrt{c}}{2}$$

$$\frac{dy}{y\sqrt{1+Y^2}} = \pm \frac{c}{2} dz$$

$$csch^{-1} Y = \pm \frac{\sqrt{c}}{2} z$$

$$Y = \pm csch \frac{\sqrt{c}}{2} z$$

$$\frac{2f}{c} = csch^2 \frac{c}{2} z$$

$$f = \frac{c}{2} csch^2 \frac{c}{2} z \rightarrow U = \frac{c}{2} csch^2 \frac{\sqrt{c}}{2} (x - ct)$$

Example 2

Conceder the following nonlinear 2nd order (1+1)-D PDE:

$$u_{tt} - u_{xx} + \sin u = 0$$

$$u(x, y) = f(x - ct) \quad , \quad \text{let } z = x - ct$$

$$u_t = -cf'$$

$$u_x = f' \quad , u_{xx} = f'' \quad , u_{xxx} = f'''$$

$$c^2 f'' - f'' + \sin f = 0$$

$$(c^2 - 1)f'' + \sin f = 0$$

$$(c^2 - 1)f'' f' + f' \sin f = 0$$

$$\frac{(c^2-1)}{2} (f')^2 + \cos f = -1$$

$$(f')^2 = (-1 + \cos f) \frac{2}{c^2-1}$$

$$(f')^2 = \frac{2}{1-c^2} (1 - \cos f)$$

$$(f')^2 = \frac{2}{1-c^2} (2 \sin^2 \frac{f}{2})$$

$$(f')^2 = \frac{4}{1-c^2} (\sin^2 \frac{f}{2})$$

$$f' = \pm \frac{2}{\sqrt{1-c^2}} \sin \frac{f}{2}$$

$$\frac{df}{\sin \frac{f}{2}} = \pm \frac{2}{\sqrt{1-c^2}} dz$$

$$2 \ln \left| \tan \frac{f}{4} \right| = \frac{2}{\sqrt{1-c^2}} z$$

$$\tan \frac{1}{4} f = e^{\frac{1}{\sqrt{1-c^2}} z}$$

$$f = 4 \tan^{-1} (e^{\frac{1}{\sqrt{1-c^2}} z})$$

$$\text{So, } u(x, t) = 4 \tan^{-1} (e^{\frac{1}{\sqrt{1-c^2}} (x-ct)})$$

Example 3

Conceder the following nonlinear 4th order (1+1)-D Boussines equation:

$$u_{tt} = u_{xx} + 3(u^2)_{xx} + u_{xxxx} \quad (10)$$

$$\text{Let } u(x, t) = f(x - ct)$$

$$u_t = -cf' \quad , \quad u_{tt} = c^2 f''$$

$$u_x = f' \quad , \quad u_{xx} = f'' \quad , \quad u_{xxx} = f''' \quad , \quad u_{xxxx} = f''''$$

$$c^2 f'' = f'' + 3(f^2)'' + f''''$$

$$c^2 f' = f' + 3(f^2)' + f'''$$

$$c^2 f = f + 3f^2 + f''$$

$$(c^2 - 1)f - 3f^2 - f'' = 0$$

$$2(c^2 - 1)ff' - 6f'f^2 - 2f'f'' = 0$$

$$(c^2 - 1)f^2 - 2f^3 - (f')^2 = 0$$

$$(f')^2 = (c^2 - 1)f^2 - 2f^3$$

$$f' = \pm\sqrt{(c^2 - 1)f^2 - 2f^3}$$

$$f' = \pm\sqrt{c^2 - 1} f \sqrt{1 - \frac{2}{c^2 - 1} f}$$

$$\text{let } y^2 = \frac{2}{c^2 - 1} f$$

$$f' = (c^2 - 1)yy'$$

$$(c^2 - 1)yy' = \pm\sqrt{c^2 - 1} \frac{c^2 - 1}{2} y^2 \sqrt{1 - y^2}$$

$$\frac{y'}{y\sqrt{1-y^2}} = \pm \frac{\sqrt{c^2-1}}{2}$$

$$\text{sech}^{-1}y = \pm \frac{\sqrt{c^2-1}}{2} z$$

$$y = \pm \text{sech} \frac{\sqrt{c^2-1}}{2} z$$

$$\sqrt{\frac{2}{c^2-1}} f = \pm \text{sec} \frac{\sqrt{c^2-1}}{2} z$$

$$\frac{2}{c^2-1} f = \text{sech}^2 \frac{\sqrt{c^2-1}}{2} z$$

$$f = \frac{c^2-1}{2} \text{sech}^2 \frac{\sqrt{c^2-1}}{2} z$$

$$u = \frac{c^2-1}{2} \text{sech}^2 \frac{\sqrt{c^2-1}}{2} (x - ct)$$

That is represent the exact solution

4. The Convergence Analysis of LADM

In this section the convergence of the suggested approach will be discuss. Now, consider the general form of equation as:

$$u - Nu = f \quad ; u \in H \quad (11)$$

where H is the Hilbert space, N is the nonlinear operator $N : H \rightarrow H$ and f is also in H. Substituting the decomposition series equations (4) and (5) in equation (11) to get:

$$\sum_{n=0}^{\infty} u_n(x, t) - \sum_{n=0}^{\infty} A_n(x, t) = f$$

So the recursive terms are got by: $u_0 = f$ and $u_{n+1} = A_n(u_0, u_1, \dots, u_n)$

Proposed approach suggest finding $B_n = u_1 + u_2 + u_3 + \dots + u_n$ by using iterative scheme

$$B_0 = 0$$

$$B_{n+1} = N(B_n + u_0), \text{ where}$$

$$N(B_n + u_0) = \sum_{i=0}^n A_i$$

If the limit exist $B = \lim_{n \rightarrow \infty} B_n$, in a Hilbert space, then B is a solution of the equation $B = N(u_0 + B)$ in H.

Theorem 1

Let N be a nonlinear operator, $N: H \rightarrow H$ where H is Hilbert space and u be the exact solution of equation (10). The decomposition series $\sum_{n=0}^{\infty} u_n(x, t)$ converges to u when

$$\exists \gamma < 1 ; \|u_{n+1}\| \leq \gamma \|u_n\| , n \in \mathbb{Z}$$

Proof

We need to prove the sequence $B_n = u_1 + u_2 + u_3 + \dots + u_n$, is a Cauchy sequence in the Hilbert space H.

$$\|B_{n+1} - B_n\| = \|u_{n+1}\| \leq \gamma \|u_n\| \leq \gamma^2 \|u_{n-1}\| \leq \dots \leq \gamma^{n+1} \|u_0\|$$

Now, we show B_n is Cauchy sequence:

$$\begin{aligned} \|B_m - B_n\| &= \|(B_m - B_{m-1}) - (B_{m-1} - B_{m-2}) - \dots - (B_{n+1} - B_n)\| \\ &\leq \|B_m - B_{m-1}\| + \|B_{m-1} - B_{m-2}\| + \|B_{m-2} - B_{m-3}\| + \dots + \|B_{n+1} - B_n\| \\ &\leq \gamma^m \|u_0\| + \gamma^{m-1} \|u_0\| + \dots + \gamma^{n+1} \|u_0\| \\ &= (\gamma^m + \gamma^{m-1} + \dots + \gamma^{n+1}) \|u_0\| \\ &\leq (\gamma^{n+1} + \gamma^{n+2} + \dots) \|u_0\| \end{aligned}$$

$$\text{So, } \|B_m - B_n\| = \frac{\gamma^{n+1}}{1-\gamma} \|u_0\|, \text{ for } n, m \in \mathbb{N}; m \geq n$$

Since $\gamma < 1$, the sequence $B_n, n = 0, \dots, \infty$ is a Cauchy sequence in the Hilbert space. Hence, $\lim_{n \rightarrow \infty} B_n = B$.

so B is the solution of equation (10)

5. The order of Convergence for suggested approach

In this section we determine the order of convergence

Definition 1 [15]

Let B_n is a sequence that converges to B. If there exist two constants p and c, $c \in \mathbb{R}, p \in \mathbb{N}$, such that

$$\lim_{n \rightarrow \infty} \left| \frac{B_{n+1} - B}{(B_n - B)^p} \right| = c$$

Then the order of convergence of B_n is p .

Proof

Consider the Taylor expansion of $N(B_n + u_0)$ around the point $(B + u_0)$, i.e.,

$$\begin{aligned} N(B_n + u_0) &= N(B + u_0) + N'(B + u_0)(B_n - B) + \frac{1}{2!} N''(B + u_0)(B_n - B)^2 + \dots + \frac{1}{m!} N^{(m)}(B + u_0)(B_n - B)^m + \dots \\ N(B_n + u_0) - N(B + u_0) &= N'(B + u_0)(B_n - B) + \frac{1}{2!} N''(B + u_0)(B_n - B)^2 + \dots + \frac{1}{m!} N^{(m)}(B + u_0)(B_n - B)^m + \dots \end{aligned} \tag{11}$$

Since $N(B + u_0) = B$ and $N(B_n + u_0) = B_{n+1}$, so equation (11) becomes

$$B_{n+1} - B = N'(B + u_0)(B_n - B) + \frac{1}{2!} N''(B + u_0)(B_n - B)^2 + \dots + \frac{1}{m!} N^{(m)}(B + u_0)(B_n - B)^m + \dots \tag{12}$$

Theorem 2

Let $N \in C^p[a, b]$ if $N^{(m)}(B + u_0) = 0$ for $m = 0, 1, 2, \dots, p-1$ and $N^{(p)}(B + u_0) \neq 0$, then the order of sequence B_n is p .

Proof

From the hypotheses of theorem, and equation (12) we get:

$$B_{n+1} - B = \frac{1}{p!} N^{(p)}(B + u_0) (B_n - B)^p + \frac{1}{(p+1)!} N^{(p+1)}(B + u_0) (B_n - B)^{p+1} + \dots \tag{13}$$

Now, dividing both sides of equation (13) by $(B_n - B)^p$ we obtain:

$$\frac{B_{n+1} - B}{(B_n - B)^p} = \frac{1}{p!} N^{(p)}(B + u_0) + \frac{1}{(p+1)!} N^{(p+1)}(B + u_0) (B_n - B) + \dots \tag{14}$$

for both sides of equation (14) we take the limit as $n \rightarrow \infty$, so we get:

$$\lim_{n \rightarrow \infty} \left| \frac{B_{n+1} - B}{(B_n - B)^p} \right| = \lim_{n \rightarrow \infty} \frac{1}{p!} N^{(p)}(B + u_0) + \lim_{n \rightarrow \infty} \frac{1}{(p+1)!} N^{(p+1)}(B + u_0) (B_n - B) + \dots$$

Since $\lim_{n \rightarrow \infty} (B_n) = B$ then we has:

$$\lim_{n \rightarrow \infty} \left| \frac{B_{n+1} - B}{(B_n - B)^p} \right| = \lim_{n \rightarrow \infty} \frac{1}{p!} N^{(p)}(B + u_0) = c$$

Then by definition 1 the order is p .

6. Conclusions

In this research we consider the inhomogeneous nonlinear (1+1) dimensional differential equations with variable coefficients. To solve it, suggested approach is applied. Then, it is seen that suggested modification has the same exact solution. In addition to their effectiveness and usefulness in solving nonlinear PDEs, we show that these decomposition methods are powerful tools in solving nonlinear (1+1) dimensional differential equations. Compared to other methods for solving this type of PDEs, there is no need for of nonlinear terms. Moreover, we can easily and rapidly attain the solution if we use suggested approach compared with ADM.

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