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ESTIMATION INTENSITY RADIATION OF CHEST X-RAY (CXR) WITH APPLICATION

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ABSTRACT

In this research we assumed that the number of emissions by time (t) of radiation particles is distributed poisson distribution with parameter (θt), where $\theta < 0$ is the intensity of radiation.

We conclude that the time of the first emission is distributed exponentially with parameter θ, while the time of the k-th emission $(k = 2,3,4, \dots\dots)$ is gamma distributed with parameters (k, θ), we used a real data to show that the Bayes estimator θ^* for θ is more efficient than $\hat{\theta}$, the maximum likelihood estimator for θ by using the derived variances of both estimators as a statistical indicator for efficiency.

KEYWORDS

Chest x-ray (CXR), emissions, intensity of radiation, Sievert, Bayes.

1. INTRODUCTION

Estimation intensity radiation for (CXR) system is very important in medical field. The technical using (CXR) systems sometimes they cannot adjusted perfectly the period time of taking (CXR) for patients or repeated it many times during a short period of time and so on the estimation intensity of radiation is very important to minimize risk infection cancer lung for the patients taking (CXR) many times for different medical purposes.

The unit of intensity radiation is the Sievert (SV) and the upper bound intensity radiation is not more than (0.06) MSV, where MSV=10^{-3} SV.

Finally, for the above reasons researchers conclude that (1.8-6) % of patients taking (CXR) infected by lung cancer [3].

2. THEORETICAL PART

A) The Distribution of the k-th Emission

The probability of emission j radiographic particles at time t is given by;

$$p(y=j)=\frac{(\theta t)^{j}e^{-\theta t}}{j!},\qquad j=0,1,2,3,\dots\dots\dots \tag{1}$$

where:

y : is random variable.
Θ : is the parameter of Poisson distribution.
t : time

$$p(x\le t)=1-p(y\le k-1)$$

$$p(x\le t)=1-\sum_{j=0}^{k-1}\frac{(\theta t)^{j}e^{-\theta t}}{j!} \tag{2}$$

where:

y : is random variable.
Θ : is the parameter of Poisson distribution.
K : natural number

The distribution of first emission;

$$p(x\le t)=1-p(y\le 0)$$

$$p(x\le t)=1-\frac{(\theta t)^{0}e^{-\theta t}}{0!}$$

$$p(x\le t)=F(t)=1-\ e^{-\theta t}\qquad \dots\dots \tag{3}$$

where:

t: is time.
Θ: is the parameter of Poisson distribution.

which is the (c.d.f) of exponential distribution with parameter θ.

The distribution of k emissions;

$$F(t)=p(x\le t)$$

$$F(t)=1-\sum_{j=0}^{k-1}\frac{(\theta t)^{j}e^{-\theta t}}{j!}$$

$$\frac{dF(t)}{dt}=-\frac{d}{dt}\sum_{j=0}^{k-1}\left[\frac{(\theta t)^{j}e^{-\theta t}}{j!}\right]$$

$$\frac{dF(t)}{dt}=-\sum_{j=0}^{k-1}\frac{d}{dt}\left[\frac{(\theta t)^{j}e^{-\theta t}}{j!}\right]$$

$$\frac{dF(t)}{dt} = -\sum_{j=0}^{k-1} \frac{(\theta t)^j\left(-\theta e^{-\theta t}\right) + e^{-\theta t}\,[j\,(\theta t)^{j-1}.\theta]}{j!}$$

$$\frac{dF(t)}{dt} = \sum_{j=0}^{k-1} \frac{\theta^{j+1} t^j e^{-\theta t}}{j!} - \sum_{j=0}^{k-1} \frac{j\,\theta^j t^{j-1}\,e^{-\theta t}}{j!}$$

Let $j = i - 1$ in the first sum.

It is clear that the second sum $= 0$ for $j = 0$ therefore we have

$$f(\mathrm{t}) = \left[\frac{\theta\,e^{-\theta t}}{0!} + \frac{\theta^2\,t\,\,e^{-\theta t}}{1!} + \cdots + \frac{\theta^{k-1} t^{k-2} e^{-\theta t}}{(k-2)!} + \frac{\theta^k t^{k-1} e^{-\theta t}}{(k-1)!}\right] - \left[\frac{\theta\,e^{-\theta t}}{1!} + \frac{2\,\theta^2\,t\,\,e^{-\theta t}}{2!} + \cdots + \frac{(k-1)\theta^{k-1} t^{k-2} e^{-\theta t}}{(k-1)!}\right]$$

$$f(\mathrm{t}) = \frac{\theta^k t^{k-1} e^{-\theta t}}{\Gamma(k)}; \quad t > 0, k > 0, \theta > 0 \quad \ldots\ldots \tag{4}$$

which is the gamma distribution with parameters (k, θ), where k known as natural number.

B) The Maximum Likelihood Estimator for $\boldsymbol{\theta}$

The likelihood function for a sample of size n is given by;

$$f(t_1, t_2, \ldots, t_n; \theta) = \frac{\theta^{nk}}{[\Gamma(k)]^n} \prod_{i=1}^{n} t_i^{k-1}\, e^{-\theta \sum_{i=1}^{n} t_i}$$

$$lnf = -nln\Gamma(k) + nkln\theta + (k-1)\sum_{i=1}^{n} lnt_i - \theta \sum_{i=1}^{n} t_i$$

$$\frac{dlnf}{d\theta} = \frac{nk}{\theta} - \sum_{i=1}^{n} t_i$$

$$\frac{d^2 lnf}{d\theta^2} = -\frac{nk}{\theta^2} < 0$$

Therefore, the critical point is maximum and hence we have;

$$\frac{nk}{\hat{\theta}} - \sum_{i=1}^{n} t_i = 0$$

$$\hat{\theta} = \frac{nk}{\sum_{i=1}^{n} t_i} \qquad \ldots\ldots\ldots\ldots\ldots \tag{5}$$

where:

k known as natural number.

C) Bayes Estimator for θ:

Using Jeffrey's prior $g(\theta) \propto \frac{1}{\theta^m}$, then the posterior density function for θ is given by [2];

$$h(\theta \backslash t_1, t_2, \dots, t_n) \propto \frac{\theta^{nk-m}}{[\Gamma(k)]^n} \prod_{i=1}^{n} t_i^{k-1} \, e^{-\theta \sum_{i=1}^{n} t_i} \quad \dots \dots \qquad (6)$$

$$h(\theta \backslash t_1, t_2, \dots, t_n) = c \, . \frac{\theta^{\,nk-m}}{[\Gamma(k)]^n} \prod_{i=1}^{n} t_i^{\,k-1} \, e^{-\theta \sum_{i=1}^{n} t_i}$$

where

$$c^{-1} = \frac{\prod_{i=1}^{n} t_i^{k-1}}{[\Gamma(k)]^n} \int_0^\infty \theta^{\,nk-m} \, e^{-\theta \sum_{i=1}^{n} t_i} \, d\theta$$

Let $\theta \sum_{i=1}^{n} t_i = s$, we get the following;

$$c^{-1} = \frac{\prod_{i=1}^{n} t_i^{k-1} \;\; \Gamma(nk-m+1)}{[\Gamma(k)]^n \left(\sum_{i=1}^{n} t_i \right)^{nk-m+1}} \quad \dots \dots \qquad (7)$$

Therefore, we have;

$$h(\theta \backslash t_1, t_2, \dots, t_n) = \frac{\left(\sum_{i=1}^{n} t_i \right)^{nk-m+1}}{\Gamma(nk-m+1)} \theta^{\,nk-m} e^{-\theta \sum_{i=1}^{n} t_i} \quad \dots \dots \dots \qquad (8)$$

which is the (pdf) of Gamma distribution with parameters $(nk-m+1, \sum_{i=1}^{n} t_i)$ using squared error loss function, the Bayes estimator θ^* for θ is the posterior mean [1].

$$\theta^* = E(\theta \backslash t_1, t_2, \dots, t_n)$$

$$\theta^* = \frac{\left(\sum_{i=1}^{n} t_i \right)^{nk-m+1}}{\Gamma(nk-m+1)} \int_0^\infty \theta^{\,nk-m+1} e^{-\theta \sum_{i=1}^{n} t_i} \, d\theta$$

$$\theta^* = \frac{\left(\sum_{i=1}^{n} t_i \right)^{nk-m+1}}{\Gamma(nk-m+1)} \cdot \frac{\Gamma(nk-m+2)}{\left(\sum_{i=1}^{n} t_i \right)^{nk-m+2}}$$

$$\theta^* = \frac{nk-m+1}{\sum_{i=1}^{n} t_i} \quad \dots \dots \dots \dots \qquad (9)$$

$$\Gamma(nk-m+2) = (nk-m+1)\Gamma(nk-m+1)$$

D) The Variance of $\hat{\propto}$ and $\propto^*$:

$$t_i \sim Gamma\ (n, \theta)$$

$$T = \sum_{i=1}^{n} t_i \sim Gamma(nk, \theta)$$

$$K(T) = \frac{\theta^{nk}}{\Gamma(nk)} T^{nk-1} e^{-\theta T} \qquad , T > 0$$

$$E\left(\frac{1}{T}\right) = \frac{\theta^{nk}}{\Gamma(nk)} \int_0^\infty \left[\frac{W}{\theta}\right]^{nk-1} e^{-W} \frac{dw}{\theta}$$

where $W = \theta T$ and $dT = \frac{dW}{\theta}$

$$E\left(\frac{1}{T}\right) = \frac{\theta}{nk-1}$$

Similarly, we have:

$$E\left(\frac{1}{T}\right)^2 = \frac{\theta^2}{(nk-1)(nk-2)}$$

$$var\left(\frac{1}{T}\right) = E\left(\frac{1}{T}\right)^2 - \left[E\left(\frac{1}{T}\right)\right]^2$$

$$var\left(\frac{1}{T}\right) = \frac{\theta^2}{(nk-1)(nk-2)} - \frac{\theta^2}{(nk-1)^2}$$

$$var\left(\frac{1}{T}\right) = \frac{\theta^2}{(nk-1)^2(nk-2)}$$

We have

$$\hat{\theta} = \frac{nk}{\sum_{i=1}^{n} t_i} = \frac{nk}{T}$$

and therefore:

$$var(\hat{\theta}) = n^2k^2 . \frac{\theta^2}{(nk-1)^2(nk-2)}$$

$$var(\hat{\theta}) = \frac{n^2k^2\,\theta^2}{(nk-1)^2(nk-2)} \qquad \dots\dots\dots \qquad (10)$$

Similarly,

$$var(\theta^*) = [nk-m+1]^2 . \frac{\theta^2}{(nk-1)^2(nk-2)}$$

$$var(\theta^*) = \frac{\theta^2[nk-m+1]^2}{(nk-1)^2(nk-2)} \qquad \dots\dots\dots\dots \qquad (11)$$

When we satisfaction $m = 1$ in equation (11) we obtain that the two estimators $\hat{\theta}$ and θ^* will having a coincide values.

3. APPLICATION PART

1. The following data represented the sum of times during the patient A and patient B taking (CXR) for many medical purposes in m minutes (m min)
   where: m minutes $= 10^{-3}\ min$

   $$\text{Patient A} = \sum_{i=1}^{10} t_i = 2000 \ \text{ m min}$$

   $$\text{Patient B} = \sum_{i=1}^{30} t_i = 3000 \ \text{ m min}$$

   We suppose a value for random parameter θ, say ($\theta = 0.05$) and correct factor for estimator, say ($m = 2$).

2. We note that if we take another values for θ and m the results will be in the same direction.

3. The emissions of radiation particles for X-ray will be ($k = 1,2, \ldots.,15$)

4. The following tables contains the estimated values of estimators $\hat{\theta}$ and θ^* and their variance for any emission $k, k \in \{1,2,3, \ldots \ldots,15\}$.

Table 1
Patient A

$\hat{\theta}$	θ^*	Var($\hat{\theta}$)	Var(θ^*)	Eff
0.00500	0.00450	0.0003858	0.0003125	0.8100000
0.01000	0.00950	0.0001539	0.0001389	0.9025000
0.01500	0.01450	0.0000955	0.0000893	0.9344444
0.02000	0.01950	0.0000692	0.0000658	0.9506250
0.02500	0.02450	0.0000542	0.0000521	0.9604000
0.03000	0.02950	0.0000446	0.0000431	0.9669444
0.03500	0.03450	0.0000378	0.0000368	0.9716327
0.04000	0.03950	0.0000329	0.0000321	0.9751563
0.04500	0.04450	0.0000291	0.0000284	0.9779012
0.05000	0.04950	0.0000260	0.0000255	0.9801000
0.00500	0.00450	0.0003858	0.0003125	0.8100000
0.01000	0.00950	0.0001539	0.0001389	0.9025000
0.01500	0.01450	0.0000955	0.0000893	0.9344444
0.02000	0.01950	0.0000692	0.0000658	0.9506250
0.02500	0.02450	0.0000542	0.0000521	0.9604000

Table 2
Patient B

$\hat{\theta}$	θ^*	Var($\hat{\theta}$)	Var(θ^*))	Eff
0.01000	0.00967	0.0000955	0.0000893	0.9344444
0.02000	0.01967	0.0000446	0.0000431	0.9669444
0.03000	0.02967	0.0000291	0.0000284	0.9779012
0.04000	0.03967	0.0000215	0.0000212	0.9834028
0.05000	0.04967	0.0000171	0.0000169	0.9867111
0.06000	0.05967	0.0000142	0.0000140	0.9889198
0.07000	0.06967	0.0000121	0.0000120	0.9904989
0.08000	0.07967	0.0000106	0.0000105	0.9916840
0.09000	0.08967	0.0000094	0.0000093	0.9926063
0.10000	0.09967	0.0000084	0.0000084	0.9933444
0.11000	0.10967	0.0000077	0.0000076	0.9939486
0.12000	0.11967	0.0000070	0.0000070	0.9944522
0.13000	0.12967	0.0000065	0.0000064	0.9948784
0.14000	0.13967	0.0000060	0.0000060	0.9952438
0.15000	0.14967	0.0000056	0.0000056	0.9955605

From Table 1 and 2 we notice that:

1. When increase the value of k the var(θ*)), var$(\hat{\theta})$ decrease.
2. The efficiency of the var(θ*) best than var$(\hat{\theta})$ for all the value of k.
3. The best efficiency when $k = 15$.

CONCLUSIONS

1. The Bayes estimator θ^* for intensity radiation of chest X-ray is more efficient than the maximum likelihood for θ because it has minimum variance for all k.
2. For large sample size of time, the probability of infection for patients by lung cancer increase.
3. According to conclusion (2) it is clear that, the infect by lung cancer has a strong relation by k the number of emissions.

   Patient A ; $k > 13$
   Patient B ; $k > 6$

4. Theoretically if the value of correct factor estimator m is one, the two estimators $\hat{\theta}$ and θ^* will having a coincide values.

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