Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 26 (2023), No. 7, pp. 1431–1437
DOI : 10.47974/JIM-1561

Nano perfect mappings

E. A. Mohsin *
Y. Y. Yousif †
Department of Mathematics
College of Education for Pure Sciences (Ibn al-Haitham)
University of Baghdad
Baghdad
Iraq

Abstract

In this paper, we will introduce and study the concept of nano perfect mappings by using the definition of nano continuous mapping and nano closed mapping, study the relationship between them, and discuss them with many related theories and results. The k-space and its relationship with nano-perfect mapping are also defined.

Subject Classification: *(2020) 54C10, 54C10.*

Keywords: *Approximation spaces, Nano topological spaces, Nano closed, Nano continuous, Nano perfect mappings.*

1. Introduction and Preliminaries

In [5] an important concept is introduced: the nano topology, which relates to a subset G of the universe U, where the components of the nano topological space are the boundary region, the upper region, and the lower region, using equivalence relations on them. [1,7] provided some definitions, namely *nm-j-ω*-perfect, *nm-j-ω*-condensation point and *nm-j-ω*-closed set. In 2021 [7], they introduced the Fibrewise soft topological spaces concept. [8] also have a role in building this work. This paper introduces nano-perfect mappings and several related theorems concerning them.

Definition 1.1: [4] Let R be a subset of the non-empty set U where they are called equivalence relation on U. Then U is said to be the universe which

* *E-mail:* **istabraq.abd2103m@ihcoedu.uobaghdad.edu.iq (Corresponding Author)**
† *E-mail:* **yousif.y.y@ihcoedu.uobaghdad.edu.iq**

divides the equivalence classes into discrete parts, where their union is a U universe and their intersection is an empty set. Let G ⊆ U. Then,

(a) The lower approximation of G is a symbol by Lo_R(G). That is, Lo_R(G) = ⋃{R(G) : R(G) ⊆ G, g ∈ U} and R(G) symboly the equivalence class determined by g ∈ U.
(b) The upper approximation of G is a symbol by Up_R(G). That is, Up_R(G) = ⋃{R(G) : R(G) ∩ G ≠ ∅, g ∈ U}.
(c) The boundary region of G is a symbol by Bo_R(G). That is, Bo_R(G) = Up_R(G) – Lo_R(G).

Definition 1.2: [5] Let R be an equivalence relation on U, then U be the universe and τ_R(G) = {U, ∅, Lo_R(G), Up_R(G), Bo_R(G)}, and G ⊆ U. Then τ_R(G) it achieves the following axioms:

(a) U and ∅ ∈ τ_R(G).
(b) The union of the components of any sub-collection of τ_R(G) is in τ_R(G).
(c) The intersection of the components of any finite subcollection of τ_R(G) is in τ_R(G).

Therefore, τ_R(G) is a topology on U is said to be the nano topology (denoted by nano-top.) on U with reference to G. We called (U, τ_R(G)) as the nano topological space (denoted by nano-top-sp.). The components of τ_R(G) are called nano-open sets. A nano closed set is the complement of a nano open set.

Definition 1.3: [5] Let (U, τ_R(G)) and (V, $\tau_{R`}$(H)) be nano-top-sp's. Then a mapping (denoted by map.) f : (U, $\tau_R$(G)) ⟶ (V, $\tau_{R`}$(H)) is nano continuous (denoted by nano-cont.) on U if the inverse image of each nano-open-set in V is nano-open-set in U.

Definition 1.4: [5] A map. f : (U, τ_R(G)) ⟶ (V, $\tau_{R`}$(H)) is a nano-closed-map if the image of each nano-closed-set in U is nano-closed-set in V.

Definition 1.5: [3] A collection {A_i : i ∈ I } of nano-open-sets in a nano-top-sp. (U, τ_R(G)) is said to be a nano-open-cover of a subset B of U if B ⊂ {A_i : i ∈ I } holds.

We go behind closely for other notations not mentioned here [2, 3, 5, 6, 7].

2. Nano Perfect Mappings

In this chapter, we will introduce nano-perfect mappings and some related theories.

Definition 2.1: A map f : (G, τ_R(G)) ⟶ (H, $\tau_{R`}$(H)) is called nano perfect if it is nano continuous, nano closed, and for each h ∈ H, $f^{-1}(h)$ is nano-compact space.

The following examples explain the definition:

Example 2.2: Let U $= \{n, m, k, e\}$ with U/R = {{n},{m} ,{k, e}}. Let $G = \{n, k\} \subset U$. Then τ_R(G) = {U, ∅ ,{n} ,{n, k, e}, {k, e}}. If V = {p, q, r, s} and $V/R^{`}$ = {{p}, {q}, {r, s}} and H = {p, r} ⊂ V. So $\tau_{R'}$(H) = {V, ∅ ,{p},{p, r, s},{r, s}}. Define f : (U, $\tau_R$(G)) ⟶ (V, $\tau_{R^{`}}$(H)) Such as : f(n) = p , f(m) = q , f(k) = r , f(e) = s . Then f is nano-perfect- map.

Example 2.3: If U = {f, g, h, o} and U/R = {{g}, {h}, {f, o}}. Let G = {f, g} ⊂ U. So τ_R (G) = {U, ∅, {g}, {f, g, o}, {f, o}}. Let V = {4, 5, 6, 7} with $V/R^{`}$ = {{4}, {5}, {6}, {7}} and H = {4, 6, 7} ⊂ V. Then $\tau_{R'}$ (H) = {V,∅ , {4, 6, 7}}. Define f : (U, $\tau_R$(G)) ⟶ (V, $\tau_{R^{`}}$(H)) as : f(f) = 4 , f(g) = 5, f(h) = 6, f(o) = 7 then f is not nano-perfect- map .

Remarks on nano perfect mapping 2.4 :

(a) Let be a nano-Hausd-sp. and f : (G, τ_R(G)) ⟶ (H, $\tau_{R^{`}}$(H)) be a nano-cont., one-to-one, map. Then, f is nano-perfect if and only if f is nano-closed.

(b) Let M ⊆ G, M be nano-Hausd-sp. and i_M : M ⟶ G be the embedding (i.e., i_M (m) = m , for all m ∈ M). Then i_M is nano-perfect if and only if NCl(M) = M.

(c) Let G be a nano-compact-sp. and H be a nano-Hausd-sp., let f : (G, τ_R(G)) ⟶ (H, $\tau_{R^{`}}$(H)) be nano-cont. So f is a nano-perfect-map.

Theorem 2.5: *If G is a nano-compact-sp. and H is nano-Hausd-sp., then the projection P : G×H ⟶ H is nano-perfect.*

Theorem 2.6: *A nano-cont. map f : (G, τ_R(G)) ⟶ (H, $\tau_{R^{`}}$(H)) is nano-closed if and only if for each point h ∈ H and all nano-open-set U ⊂ G such as f^{-1}(h) ⊂ U, then find an nano-open-set V of H such as h ∈ V and f^{-1}(V) ⊂ U.*

Theorem 2.7: *If f : (G, τ_R(G)) ⟶ (H, $\tau_{R^{`}}$(H)) is a nano-perfect-map, then for each nano-compact-sub-sp. I ⊂ H the inverse image f^{-1}(I) is nano-compact.*

Corollary 2.8: *The composite of nano-perfect- maps is a nano-perfect map.*

Lemma 2.9: *If A is a nano-compact-sub-sp. of a nano-regu-sp. G, then for each nano-compact-sub-sp. B disjoint from A find nano-open-sets U, V ⊂ G such as A ⊂ U, B ⊂ V and U ∩ V = ∅.*

Lemma 2.10: *A nano-perfect-map f : (G, τ_R(G)) ⟶ (H, $\tau_{R^{`}}$(H)) cannot be nano-continuously extended over any nano-Hausd-sp. I consist of G as a nano-dense perfect-sub-space.*

Proposition 2.11: If the structure of nano-cont. Map f : (G, τ_R(G)) ⟶ (H, $\tau_{R^{`}}$(H)) and j : (H, $\tau_{R^{`}}$(H)) ⟶ (I, τ_{R^*}(I)), where H is a nano-Hausd-sp., is nano-perfect, then the map j $\big|_{f(g)}$ and f are nano-perfect.

Theorem 2.12: *If f : (G, τ_R(G)) ⟶ (H, $\tau_{R`}$(H)) is a nano-perfect-map., then for any nano-closed A ⊂ G and any B ⊂ H the restrictions $f|_A$: A ⟶H and f_B: f^{-1}(B) ⟶B are nano-perfect.*

Theorem 2.13: *Let f : (G, τ_R(G)) ⟶ (H, $\tau_{R`}$(H)) be a nano-perfect-map. If G is nano-dense in a nano-Hausd-space. G_1, and $H \cap H_1$, then any nano-cont. extension f_1 : G_1 ⟶ H_1 of f maps $G_1 - G$ into $H_1 - H$.*

Proof: Suppose not. Then there is a g ∈ G_1– G with f_1(g) ∈ H. Let h = f_1(g) and A = f^{-1}(h). Now A is nano-compact and g ∉ A, so there are disjoint nano-open U and V in G_1 with g ∈ U and A ⊂ V. Then g ∈NCl(U⋂G)⊂NCl(G-V), so h = f_1(g) ∈ H ⋂ NCl f_1(G-V) = H ⋂ NCl f(G-V) = f(G-V) which is contradiction. That completes the proof.

Theorem 2.14: *Let G be a nano-Hausd-sp., a finite cover $\{A_i\}_{i=1}^{k}$ of the space G and a family $\{Nf_i\}_{i=1}^{k}$ of appropriate maps, where Nf_i : A_i ⟶ Y, such as the combination $Nf = Nf_1 \nabla Nf_2 \nabla \dots \nabla Nf_k$ is nano-cont. If all maps Nf_i are nano-perfect, then the combination f is nano-perfect.*

Lemma 2.15: *If $\{A_s\}_{s\in S}$ is a locally finite nano-closed-cover of a space G and $\{f_s\}_{s\in S}$ where f_s : A_s ⟶H, is a family of appropriate nano-cont. map, then the combination f = $\nabla_{s\in S} f_s$ is a nano-cont. map of G to H.*

Theorem 2.16: *If $\{A_i\}_{i=1}^{k}$ is a finite nano-closed cover or a finite nano-open cover of a nano-Hausd-sp. G and $\{f_i\}_{i=1}^{k}$ where f_i : A_i ⟶H, is a family of appropriate nano-perfect-maps, then the combination f = $f_1 \nabla f_2 \nabla \dots \nabla f_k$ is a nano-perfect-map of G to H.*

Theorem 2.17: *The Cartesian product f = $\prod_{s\in S} f_s$ where f_s : X_s ⟶ Y_s and $X_s \neq \emptyset$ for s ∈ S, is nano-perfect if and only if all mappings f_s are nano-perfect.*

Theorem 2.18: *The diagonal of any family of nano-perfect-maps is nano-perfect-map.*

Proof: The diagonal can be represented as the restriction of the Cartesian product of maps to a nano-closed-set.

Let us note the fact that the diagonal. $f_1 \Delta f_2$ is a nano-perfect-map does not imply that f_1 and f_2 are nano-perfect, as the following example shows :

Example 2.19: Let U_1 = {a, b, c} with U_1/R= {{a}, {b}, {c}} and G = {a, b} ⊂ U_1 then τ_R(G) = {U_1, ∅, {a, b}}. Let V_1= {1, 2, 3} and $V_1/R`$ = {{1}, {2}, {3}} and H = {1, 2} ⊂ $V_1$ then $\tau_{R`}$(H) = {V_1, ∅, {1, 2}}. Define f_1: (U_1, τ_R (G)) ⟶ (V_1, $\tau_{R`}$(H)) such as : f(a) = 1, f(b) = 3, f(c) = 2 then f_1 is not nano-perf-map. because of f_1 is not nano-cont. Mapping and not nano-closed mapping.

Let U_2 = {d, e, f} with U_2/R = {{d}, {e}, {f}} and G = {d, e} ⊂ U_2 then τ_R(G) = {U_2, ∅, {d, e}}. Let V_2 = {4, 5, 6} and $V_2/R^`$ = {{4}, {5}, {6}} and H = {4, 5} ⊂ $V_2$ then $\tau_{R^`}$(H) = {V_2, ∅, {4, 5}}. Define f_2 : (U_2, τ_R(G)) ⟶ (V_2, $\tau_{R^`}$(H)) such as : f(d) = 4, f(e) = 6, f(f) = 5 then f_2 is not nano-perf-map. because of f_2 is not nano-cont. Mapping and not nano-closed mapping.

Let U = {a, b, c} with U ∕ R = {{a}, {b}, {c}} and G = {a, b} ⊂ U then τ_R(G) = {U, ∅, {a, b}}. Let V_1 = {1, 2, 3} and V_2 = {4, 5, 6} and $V_1 \times V_2$ = {{1} × V_2 , {2} × V_2 , {3}× V_2} and $(V_1 \times V_2)/R^`$ = {{1} × $V_2$ , {2} × $V_2$ , {3}× $V_2$} and H = {{{1} × $V_2$ , {2} × $V_2$} ⊂ $V_1 \times V_2$ then $\tau_{R^`}(V_1 \times V_2)$ = {$V_1 \times V_2$, ∅, {1} × V_2 , {2} × V_2}. Define Δ : (U, τ_R(G)) ⟶ ($V_1 \times V_2$, $\tau_{R^`}(V_1 \times V_2)$) such as : f(a) = {1} × V_2 , f(b) = {1} × V_2, f(c) = {3}× V_2 then Δ is nano-perf-map.

It turns out that the last theorem can be significantly strengthened.

Theorem 2.20: *Suppose we are given a family of nano-cont.maps* $\{f_s\}_{s\in S}$ *, where* f_s *: G⟶* H_s*. If there an* $s_o \in S$ *such as* f_{s_o} *is a nano-perfect-map and* H_s *is a nano-Hausd- sp. for each s ∈* $S\backslash\{s_o\}$*, then the diagonal* $\Delta_{s\in S} f_s$ *is a nano-perfect- map.*

Corollary 2.21: *If G admits a nano-perfect-map into a top- sp.H and a nano-cont. one-to-one map into a nano-Hausd-sp. I, then G, is homeomorphic to a nano-closed-sub-sp. of H×I.*

Corollary 2.22: *If H is any nano-top- sp., then the following properties of a completely nano-regu-sp. G are equivalent :*

(a) A nano-perfect-map f : (G, τ_R*(G)) ⟶ (H,* $\tau_{R^`}$*(H)).*
(b) G is homeomorphic to a nano-closed-sub-sp. of H×I for some nano-compact nano-Hausd-sp. I.
(c) G is homeomorphic to a nano-closed-sub-sp. of H×I for some nano-compact-sp. I.

This corollary is an analogue of Proposition 2.11, so we give another proof of the second part of this theorem.

Corollary 2.23: *Let f : (G,* τ_R*(G)) ⟶ (H,* $\tau_{R^`}$*(H)) and j : (H,* $\tau_{R^`}$*(H)) ⟶ (I,* τ_{R^*}*(I)) be nano-cont., and suppose that j is nano-perfect and H is nano-Hausd. Then f is nano-perfect.*

Corollary 2.24: *If t : (G,* τ_R*(G)) ⟶ (I,* τ_{R^*}*(I)) is nano-perfect and has a nano-cont. Extension j : (H,* $\tau_{R^`}$*(H)) ⟶ (I,* τ_{R^*}*(I)) for some nano-Hausd- sp. H ⊃ G, then G is nano-closed in H.*

Proof: Let f : (G, τ_R(G)) ⟶ (H, $\tau_{R^`}$(H)) be the injection mapping. Now, consider j : (G, $\tau_R$(G)) ⟶ (I, $\tau_{R^*}$(I)). Since t = j, so by Corollary 2. 24, f : (G, $\tau_R$(G)) ⟶ (H, $\tau_{R^`}$(H)) is nano-perfect, implying that f(G) = G ⊂ H is nano-closed.

The Kuratowski and Mrowka Theorem For Nano-Top-sp. 2.25 :
For a nano-Hausd-sp G the statement are equivalent :
(a) Space G is nano-compact.
(b) For each nano-top-sp H, the proj. P : G×H ⟶ H is nano-closed.
(c) For each nano-normal space H the projection P :G×H ⟶ H is nano-closed.

The following interesting characterization of nano-perfect-maps also is connected with Theorem 2.18.

Theorem 2.26: *For a nano-cont. map f: (G, τ_R(G)) ⟶ (I, τ_{R^*}(I)) defined on a nano-Hausd-sp G the statement is equivalent:*

(a) The map f is nano-perfect.
(b) For each nano-Hausd-sp H the Cartesian product f×id_H is nano-perfect.
(c) For each nano-Hausd-sp H the Cartesian product f×id_H is nano-closed.

In the realm of nano Tychon off-spaces the class of nano-perfect-maps can be characterized in terms of extensions. We shall now give two characterizations of this kind; in the first of these characterizations, we assume that the spaces G and H are subspaces of their compactifications. Considered in the theorem.

Theorem 2.27: *For a nano-cont. map f : (G, τ_R(G)) ⟶ (H, $\tau_{R`}$(H)), where G and H are nano-Tychonoff-space, the statement are equivalent :*

(a) The map f is nano-perfect.
(b) For each nano-compactification αH the extension F_α: $\beta G \longrightarrow \alpha H$ of the map f satisfies the condition $F_\alpha(\beta G\backslash G) \subset \alpha H\backslash H$.
(c) The extension F: $\beta G \longrightarrow \beta H$ of the map f satisfies the condition F($\beta G\backslash G$) $\subset \beta H\backslash H$.
(d) Find a nano-compactification αH such as the extension. F_α : $\beta G \longrightarrow \alpha H$ of the map f satisfies the condition $F_\alpha(\beta G\backslash G) \subset (\alpha H\backslash H)$.

Theorem 2.28: *A nano-cont. f : (G, τ_R(G)) ⟶ (H, $\tau_{R`}$(H)), where G and H are nano-Tychonoff-spaces, is nano-perfect if and only if it cannot be nano-continuously extended over any nano-Hausd-sp I that consist of G as a nano-dense perfect-sub-sp.*

Proposition 2.29: Let f : (G, τ_R(G)) ⟶ (H, $\tau_{R`}$(H)) be a nano-perfect-map from a space G onto H, and let S be a nano-dense subset of G. The nano-cont. map j: S ⟶f(S) of S onto f(S) given by the restriction of f is a nano-perfect-map if and only if S = f^{-1}f(S).

Definition 2.30: In a nano-top-sp. G is a nano-Hausd-compactly-generated-sp. (denoted by nano k-sp.) in which a subset is nano-closed if an intersection with any nano-compact-sub-set is closed.

References

[1] G. S. Ashaea and Y. Y. Yousif, Some Types of Mappings in Bitopological Spaces, *Baghdad Science Journal*, University of Baghdad-Collage of Science for Woman, Vol. 18, No.1, pp. 149-155, (2021).

[2] N. Bourbaki, General Topology, Part I, Addison Wesley, Reading, Mass, (1996).

[3] S. Krishnaprakash, R. Ramesh and R. Suresh, Nano Compactness and Nano Connectedness in Nano Topological space, *International Journal of Pure and Applied Mathematics,* Vol. 119, no.13, pp. 107–115, (2018).

[4] M. L. Thivagar and C. Richard, On nano continuity, *International Journal of Mathematics and Statistics Invention*, 1(1), 31–37, (2013).

[5] M. L. Thivagar and C. Richard, On Nano Continuity, *Mathematical Theory and Modelling*, vol. 3, no.7, pp. 32–37, (2013).

[6] A. C. Upadhya and Mamata M. K., On Nano Gp-Regular and Nano Gp-Normal Spaces, The International *Journal of Analytical and Experimental Modal Analysis*, Karnataka, India, ISSN NO: 0886-9367, p:3770-3775, July (2020).

[7] Y. Y. Yousif, L. A. Hussain & M. A. Hussain, Fibrewise soft bitopological spaces, *Journal of Interdisciplinary Mathematics*, 24:7, 1925-1934 (2021), DOI: 10.1080/09720502.2021.1966948.

[8] Y. Y. Yousif, Fibrewise ω-compact and locally ω-compact spaces, *Journal of Interdisciplinary Mathematic*s, 24:7, 1863-1869, (2021) DOI: 10.1080/09720502.2021.1964730.