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RELATIVE QUASI- INVERTIBLITY

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ABSTRUCT

The purpose of this paper is to investigate the concept of relative quasi-invertible submodules motivated by rational submodules and quasi-invertible submodules. We introduce several properties and characterizations to relative quasi-invertiblity. We further investigate conditions under which identification consider between rationality, essentiality and relative quasi-invertiblity. Finally, we consider quasi-invertiblity relative to certain classes of submodules.

مقاسات جزئية شبه عكوسة نسبياً

الخلاصة

في هذا البحث هدفنا هو استقصاء مفهوم المقاسات الجزئية النسبية الشبه عكوسة مستندين بذلك على مفهومي المقاسات الجزئية الشبه عكوسة والمقاسات الجزئية الراشدة. تم عرض جملة من التوصيفات والخواص لهذا النوع من المقاسات الجزئية، فضلا عن ذلك عرضنا الشروط التي توحد بين المقاسات الجزئية النسبية الشبه عكوسة والمقاسات الجزئية الجوهرية،المقاسات الجزئية الراشدة والمقاسات الجزئية العكوسة .
واخيرا تأملنا المقاسات الجزئية الشبه عكوسة نسبة لانماط معينة من المقاسات الجزئية.

1. Introduction:

Thought this paper R represents a commutative ring with non-zero identity and R-modules are left until. Let $N\subseteq M$ be R-modules and $y\in M$. Then:

$[N:y]=\{r\in R : ry \in N\}$ is an ideal of R. A non-zero submodule N of an R-module M is essential if $[N:y]\neq 0$ for each non-zero element $y\in M$, that is for each non-zero element $y\in M$ there exists $r\in R$ such that $r y\in N$ and $r y \neq 0$.

The concept of rational submodules was introduced originally in [1] to construct the maximal left ring of quotients. A non-zero submodule N of an R-module M is rational if $[N:y]x\neq 0$ for each $x,y\in M$ with $x\neq 0$. Equivalently, for each $x,y\in M$ with $x\neq 0$,there is $r\in R$ such that $ry\in N$ and $rx \neq 0$. Clearly, rational submodules are refinement of essential submodules. It is known that, N is rational in M if and only if $Hom_R(M/N, E(M))=0$ where *E(M)* is the injective envelope of M. This is equivalence to saying that for each submodule K of M with $N\subseteq K \subseteq M$, every R- homomorphism $\varphi: K \to M$ with $N \subseteq \ker(\varphi)$ is zero [2].

A submodule N of an R-module M is called quasi-invertible if $Hom_R(M/N, M) = 0$. Quasi-Dedekind R-modules are those in which all non-zero submodules are quasi-invertible [3] By a uniform R-module we mean a non-zero module in which every non-zero submodule is essential. An R-module M is called monoform (sometimes termed as strongly uniform) if M is non-zero and every non-zero submodule is rational in M. Thus it is clear that every monoform R-module is uniform. Since every rational submodule is quasi-invertible, then every monoform R-module is quasi-Dedekind.

Given any R-module M. An R-module X is called M- injective if for each submodule N of M and R-homomorphism α: N→X, there is an R-homomorphism β:M→X which extends α [4]. An R-module M is quasi-injective if M is M-injective [5]. An R-module M is called multiplication if each submodule of M is of the form AM for some ideal A of R [6]. An R-module M is multiplication if and only if N=[N:M]M for every submodule N of M [7].The concept of weak multiplication modules based on the last characterization was introduced in [8]. An R-module M is called weak multiplication if N=$[N:M]'$M for each submodule N of M where:

$$[N:M]'=\{\alpha\in End(M):\alpha M\subseteq N\}$$

Let M be an R-module. In this paper we introduce the concept of quasi-invertible submodule N relative to a submodule P of N in M. Quasi-invertible submodules of M are quasi-invertible relative to M it self .And rational submodules of M are quasi-invertible relative to a submodules of M. Thus quasi-invertible submodules and rational submodules motivate us to consider relative quasi-invertible submodules. Several characterizations are given, and we use these characterizations to investigate the properties of relative quasi-invertible submodule analogous to those of rational and essential submodules. Conditions are suggested to relate the above three concepts. The following is some of our results, it is shown under multiplication modules with prime annihilator (resp. non-singular quasi-injective module), a submodule N is quasi-invertible relative to submodule P if and only if N is essential in P if and only if N is rational in P. Also we study the quasi-invertible relative to multiplication (resp. weak multiplication, projective, regular and stable) submodules. It was shown that every quasi-invertible submodule N of an R-module M relative to multiplication submodule P of M with N⊂P is rational in P.

2. Relative quasi-invertible submodules:

In this section we introduce certain quasi-invertibility condition in terms submodules.

(2.1) Definition:

Let M be an R-module. A proper submodule N of M is called quasi-invertible relative to a submodule P of M if P contains N properly and Hom_R (P/N, M) =0. An R-module M is called quasi-monoform if each non-zero proper submodule of M is quasi-invertible relative to some submodule of M.

The following proposition provides two characterizations of relative quasi-invertible submodules.

(2.2) Proposition:

Let N be a proper submodule of an R-module M. Consider the following conditions.

1. N is a quasi-invertible relative to a submodule P of M with N⊂P,
2. Every R-homomorphism f: P→M with f(N) =0 is trivial,
3. For each m_1∈P, m_2 ∈M with $m_2\neq 0$, there exist r∈R such that rm_1∈N and $rm_2\neq 0$ (that is [N: m_1] $m_2\neq 0$ for each m_1∈P and m_2∈M with $m_2\neq 0$).

Then (3) ⇒(2)⇔(1). In additional, if M is P-injective module, then (2) ⇔ (3).

Proof:

(1) ⇒ (2) It is clear.

(2) ⇒ (3): Assume that there exists m_1∈P and m_2∈M with m_2≠0 such that for each r∈R, rm_1∈N implies that rm_2=0.Define θ:N+Rm→M by θ(x+rm_1)=rm_2 for each x∈ N. Then θ is well defined non-zero R-homomorphism with N⊆ ker(θ). P-injectivety of M implies that θ can be extended to an R-homomorphism

$\psi : P \longrightarrow M$ and N$\subseteq$ $\ker(\psi)$ which contradicts (2).

(3)⇒ (2): Assume that there exists a non-zero R-homomorphism $\alpha:P\to M$ with $N\subseteq \ker(\alpha)$, then there exists $m_1\in P$ with $\alpha(m_1)=m_2\in M$ and $m_2\neq 0$. Since $rm_1\in N$ implies $rm_2=0$ for all $r\in R$ which is a contradiction.

(2.3) Examples and Remarks:

1. The zero submodule is not quasi-invertible relative to all submodules P in M, otherwise M=0.
2. If N is a quasi-invertible submodule of an R-module M relative to submodules P of M with $N\subset P$, then N is not a direct summand of P.
3. If N is quasi-invertible relative to all asubmodules P of M with $N\subset P$, then $ann_R(N) = ann_R(P)$. For, let $r \in ann_R(N)$, Define $\alpha:P/N\to M$ by $\alpha(x+N)=rx$ for each $x\in P$. It is clear that α is well defined R-homomorphism. Thus $\alpha=0$ and hence $r\in ann_R(P)$.The converse is not true in general, in the Z-module $Z\oplus Z$, $ann_R(0\oplus Z) = ann_R(Z\oplus 2Z)$ but $0\oplus Z$ is not quasi-invertible relative to $Z\oplus 2Z$.

Recall that an R-module M is called prime if $ann_R(N) = ann_R(M)$ For each non-zero N of M [9].Thus if P is a submodule of M such that every non-zero submodule of P is quasi-invertible relative to P, then P is a prime module.

4. It is clear that if N is a quasi-invertible submodule of an R-module M, then it is relative quasi-invertible, in fact N is quasi-invertible relative to M itself. Thus every quasi-Dedekind R-module is quasi-monoform. Further, as every rational submodule of an R-module is quasi-invertible, thus every monoform module is quasi-Dedekind. In quasi-monoform R-module, every maximal submodule is quasi-invertible.
5. As we have mentioned in the introduction that, a submodule N of an R-module M is rational if and only if for each $m_1,m_2\in M$ with $m_2 \neq 0$ there is $r\in R$ such that $rm_1\in N$ and $rm_2\neq 0$. Note that the above elements-charactirization of rational submodules implies condition (3) in proposition (2.2), thus every rational submodule of an R-module is relative quasi-invertible. However the converse may not be true in general, consider $M=Q\oplus Z_2$ as Z-module and $N=3Z\oplus Z_2$. Let $y= (1/2,\bar{0})\in M$, $x=(0,\bar{1})\in M$. If $ry\in M$, then r is even integer, so rx=0, this implies that N is not rational in M. If $P=Z\oplus Z_2$, then for each $(n,\bar{x})\in P$, and $(p/q,\bar{y})\in M$ with $p\neq 0$ or y=1. Thus $3(n,\bar{x})\in N$ and $3(p/q,\bar{y})\neq 0$. Therefore N is quasi-invertible relative to a submodule P of N in M, but N is not rational in M.
6. It follows from proposition (2.2), if a submodule N is quasi-invertible relative to a submodule P of an R-module with $N\subset P$, and M is P-injective, then N is rational (hence essential) in P.
7. If N is quasi-invertible relative to a submodule P of M with $N\subset P$ and M is P-injective, then N is quasi-invertible relative to P in each essential extension W of M (hence in E(M)). For, let $m_1\in P$ and $m_2\in W$ with $m_2\neq 0$. As W essential extension of M, there exists $r\in R$ such that $rm_2\in M$ with $rm_2\neq 0$, further $rm_1\in P$ Proposition (2.2) implies that there is an element $t\in R$ such that $(t\,r)m_1\in N$ and $(t\,r)\,m_2\neq 0$.

In the following propositions we provide some properties of relative quasi-invertible submodules.

(2.4) Proposition:

1. Let $N_i\subset P_i\subseteq M$ and M is P_i-injective i=1,2… n. If N_i is quasi-invertible

relative to P_i, then $\bigcap_{i=1}^{n} N_i$ is quasi-invertible relative to $\bigcap_{i=1}^{n} P_i$.

2. Let $N \subset P \subset Q \subseteq M$ and M is Q-injective. Then N is quasi-invertible relative to P and P is quasi-invertible relative to Q if and only if N quasi-invertible relative to Q.

Proof:

1. Let $x \in \bigcap_{i=1}^{n} P_i$ and $y \in M$ with $y \neq 0$. By proposition (2.2) there exists $r_1 \in R$ such that $r_1 x \in N_1$ and $r_1 y \neq 0$.Also there exists $r_2 \in R$ such that $(r_2\ r_1)\ x \in N_1 \cap N_2$ and $(r_2\ r_1)\ y \neq 0$. After a finite number of steps, there is $r_n \in R$ such that $(r_n \ldots r_2 r_1)x \in \bigcap_{i=1}^{n} N_i$ and $(r_n \ldots\ldots r_2\ r_1)\ y \neq 0$. This completes the proof of (1).
2. It is enough to show the "if" part. Let $m_1 \in Q$ and $m_2 \in M$ with $m_2 \neq 0$. By proposition (2.2) there is $r \in R$ such that $rm_1 \in P$ and $rm_2 \neq 0$.Remark (2.3) (6) implies that P is essential in Q, so there exists $s \in R$ with $(sr)m_1 \in P$ and $(sr)m_2 \neq 0$.Finally there is $t \in R$ such that $(tsr)m_1 \in N$ and $(tsr)m_2 \neq 0$.

The following corollary follows from proposition (2.4) and proposition (1.12) in [5].

(2.5) Corollary:

Let M be a quasi-injective R-module. Then:

1. If N_i is quasi-invertible relative to a submodule P_i of M with $N_i \subset P$, i=1,2…n, then $\bigcap_{i=1}^{n} N_i$ is quasi-invertible relative to $\bigcap_{i=1}^{n} P_i$.
2. Let $N \subset P \subset Q \subseteq M$. Then N is quasi-invertible relative to Q if and only if N is quasi-invertible relative to P and P is quasi-invertible relative to Q.
3. Every quasi-invertible submodule relative to a quasi-invertible submodule is quasi-invertible.

(2.6) Proposition:

Let M_1, M_2 be R-modules and let $f:M_1 \to M_2$ be R-homomorphism, if N is quasi-invertible relative to all submodules P of M_2 with $N \subset P$, Then $f^{-1}(N)$ is quasi-invertible relative to $f^{-1}(P)$ in M_1.

Proof:

Let $m_1 \in f^{-1}(P)$ and $m_2 \in M_1$ with $m_2 \neq 0$. Then $f(m_1) \in P$ and $f(m_2) \in M_2$ with $f(m_2) \neq 0$. By Proposition (2.2), there exists $r \in R$ such that $rf(m_1) \in N$ and $rf(m_2) \neq 0$. This implies that $rm_1 \in f^{-1}(N)$ and $rm_2 \neq 0$ and hence $f^{-1}(N)$ is quasi-invertible relative to $f^{-1}(P)$ in M_1.

(2.7) Corollary:

Let M be an R-module and N be a proper submodule of E(M).If N is quasi-invertible relative to a submodule P of E(M) with $N \subset P$ then $N \cap M$ is quasi-invertible relative to a submodule $P \cap M$ of M with $N \cap M \subset P \cap M$.

Like remark (2.3)(6), in the following theorem under the multiplication property of the module we relate relative quasi-invertible submodules and rational submodules.

(2.8) Theorem:

Let M be a multiplication R-module. If N is quasi-invertible relative to a submodule P of M with $N \subset P$, then N is rational in P.

Proof:

Let L be a submodule of P with $N \subseteq L$ and $\varphi: L \to P$ be an R-homomorphism with $N \subseteq \ker(\varphi)$. Suppose that $\varphi \neq 0$. Then there exists a non-zero element $x \in L$ with $\varphi(x) \neq 0$.L=AM for some ideal A in R. Thus $x = \sum_{i=1}^{n} a_i m_i$ where $a_i \in A$ and $m_i \in M$. There exists j, $1 \leq j \leq n$ such that $a_j m \neq 0$ and $\varphi(a_j m_j) \neq 0$. Define $\Psi: P \to L$ by $\Psi(p) = a_j p$ for each p∈P. Thus $\varphi \circ \Psi : P \to M$ with N ⊆ker($\varphi \circ \Psi$) and $\varphi \circ \Psi \neq 0$. This contradicts condition (2) in proposition (2.2) and hence N is ratinal in P.

(2.9) Theorem:

Let M be a multiplication R-module with prime annihilator in R and N, P be submodules of M with N⊂P. If N is essential in P, then N is quasi-invertible relative to a submodule P of M with N⊂P.

Proof:

Assume there is a non-zero R-homomorphism α : P/N→M. Thus there exists an element m∈P\N such that α (m+N)=x(≠0)∈M. Hence rm(≠0)∈N for some non-zero r∈R. Thus rx=rα (m+N)=0 implies that r∈ $ann_R(x)$.

As M is multiplication, then Rx=AM for some an ideal A of R. Hence rAm=0 and rA⊆ $ann_R(M)$. Primeness of $ann_R(M)$ implies that either r∈ $ann_R(M)$ or A⊆ $ann_R(M)$. If r∈ $ann_R(M)$, then rm=0 which is a contradiction. If A⊆ $ann_R(M)$, then Rx=0 and hence x=0 which is again a contradiction. This completes the proof.

(2.10) <u>Corollary</u>:

Let M be a faithful multiplication module over integral domain D and N⊂P⊆M be submodules of M. If N is essential in P, then N is quasi-invertible relative to P.

(2.11) Corollary:

Let M be a multiplication R-module with prime annihilator in M and N⊂P⊆M be submodules of M. Then the following are equivalent.

1. N is quasi-invertible relative to P.
2. N is rational in P.
3. N is essential in P.

Proof:

(1)⇒(2) follows from theorem(2.8).

(2) ⇒(3) trivial. **(3)⇒(1)** follows from theorem (2.9).

(2.12) Proposition:

Let M be a multiplication R-module with prime annihilator in R and P be a uniform submodule of M. Then every non-zero submodule of M contained properly in P is quasi-invertible relative to P.

Proof:

Let N be a non-zero submodule of M with N⊂ P. Then N is essential submodule in of P. Theorem (2.9) implies that N is quasi-invertible relative to P in M.

Proposition (2.12) assert that every uniform multiplication R-module with prime annihilator in R. is quasi-monoform. On the other hand F.Mehdi in [10] introduced multiplication modules as follows: An R-module M is called multiplication if for every pair of submodules L and N of M with L⊂ N, there exists an ideal A of R such that L=AN. It is clear that every multiplication R-module (in the sense of F. mehdi) is multiplication and all it's submodules are multiplication. Again by using Proposition (2.12) we have the following: Let M be a prime multiplication R-module (in the sense of F- mehdi). Then every uniform proper submodule of M is quasi-monoform.

Now we consider another specter to relate relative quasi-invertibility with essential (rational) property.

(2.13) <u>Theorem</u>:

Let M be a non-singular R-module, N⊂P be submodules of M and M is P-injective. Then the following are equivalent:

1. N is quasi-invertible relative to P.
2. N is rational in P.
3. N is essential in P.

Proof:

The equivalence between (2) and (3) follows from the fact that Z(K)=Z(M)∩K for each submodule K of M. (1)⇒(2) follows from remark (2.3)(6). (2)⇒(1): Let f: P/N→M be a non-zero R-homomorphism. Then there is an element x∈P\N with f(x+N)=m∈M and m≠0.Let r∈R with r$\notin ann_R(m)$ Then rm≠0 and hence rx∉N. But N is essential in P, then there is an element t∈R with (tr)x∈N and (tr)x≠0.Thus 0= $f(tr\bar{x})$ = trm and hence tr∈ $ann_R(m)$. This implies that $ann_R(m)$ is essential ideal of R. Since M is non-singular, then m=0 which is a contradiction.

(2.14) Corollary:

Let M be a non-singular quasi-injective R-module. Then the following are equivalent for a proper submodule N of M.

1. N is quasi-invertible relative to a submodule P of M with N⊂P.
2. N is rational in P.
3. N is essential in P.

In the following we relate relative quasi-invertible submodules with rational submodules in case certain type of submodules.

(2.15) Theorem:

Let N be a proper submodule of an R-module M. If N is quasi-invertible relative to a weak multiplication submodule P of M with N⊂P and N is fully invariant in P, then N is rational in P.

Proof:

Let L be a submodule of P with N ⊆L and there exists a non-zero R-homomorphism α :L→P with N ⊆ker(α). Then there is an element t∈L with α (t)≠0.Since P is weak multiplication, then L=$\sum\Phi(P)$ where the sum runs over all endomorphism of P with $\Phi(p)$⊆L. Thus t $=\sum_{i=1}^{n}\Phi_i(p_i)$ where Φ_i :P→L and p_i∈P. So there exists j, $1\leq j\leq n$ Such that $\Phi_j(p_j)\neq 0$ and $\alpha\circ\Phi_j(p_j)\neq 0$. Let h=$\alpha\circ\Phi_j$ .Then h:P→P is a non-zero R-homomorphism with N ⊆ ker(h), since N is fully invariant in P. Define w: P/N→M by w(p+N)=h(p) for each p∈P. w is a non-zero well-defined R-homomorphism and hence $Hom_R(P/N,M)\neq 0$ which contradicts the relative quasi-invariability of N. Thus α =0 and N is rational in P.

It is known that the class of weak multiplication modules contains the classes of multiplication, projective and regular modules [8]. Then we have the following corollaries.

(2.16) Corollary:

Let N be a proper submodule of an R-module M. If N is quasi-invertible relative to projective (regular) a submodule P of M withN⊂P and N is fully invariant in P, then N is rational in P.

It is well known that every submodule of multiplication R-module is fully invariant.

(2.17) Corollary:

Let M be an R-module. Every quasi-invertible submodule N of M relative to a multiplication submodule P of M with N⊂P is rational in P.

Recall that a submodule N of an R-module M is stable if θ(N) ⊆N for each R-homomorphism θ:N→M. An R-module M is called fully stable if each submodule of M is stable [11].

(2.18) Theorem:

Let M be an R-module. Then the following are equivalent for submodules N and P of M with N⊂P and P stable.

1. N is quasi-invertible relative to P.

2. For each R-homomorphism Φ :N $\to$P, if Φ has an extension $\Psi \in End(P)$, then Ψ is unique.

3. Each α $\in$End(P) with $\alpha|_N$ is the inclusion map of N into P is the identity R-homomorphism.

Proof:

(1)⇒(2): Assume there are two extension $\Psi_1, \Psi_2 \in End(P)$ such that $\Psi_1(n) = \Psi_2(n) = \Psi(n)$ for each n$\in$ N. Define f: P/N$\to$M by f(x+N)= $\Psi_1(x) - \Psi_2(x)$ for each x$\in$P. It is clear that f is well-defined homomorphism. By (1), f=0 and hence $\Psi_1 = \Psi_2$.

(2)⇒(3): Clear.

(3)⇒(1): Let f:P/N$\to$M. Then $f \circ \pi : P \to M$ where π is the natural homomorphism of P onto P/N. Stability of P implies that $f \circ \pi \in End_R(P)$ .Let $g = I_P - f \circ \pi$. Hence $g(N) = N - f \circ \pi(N) = N$, so g is the inclusion map of N into P. By (3), $g = I_P$ and this implies that $f \circ \pi = 0$,hence f=0.

It was proved in [11] that every fully stable R-module over a Dedekind domain is quasi-injective. Then we have the following corollary.

(2.19) <u>Corollary</u>:

Let M be a fully stable module over a Dedekind domain D. Then the following are equivalent for a proper submodule N of M.

1. N is quasi-invertible submodule.
2. N is quasi-invertible relative to each submodule P of M with N$\subset$P .
3. Each R-homomorphism Φ :N $\to$P has a unique extension to P.
4. Each $\alpha \in End_R(P)$ with $\alpha|_N = I_N$ is the identity R-homomorphism.

Proof :

(1)⇒(2) : N is quasi-invertible relative to a submodule M. Let P be any submodule of M with N$\subset$P. As M is quasi-injective, then N is quasi-invertible to P, proposition (2.4) (2).

(2) ⇒(1): Clear. The equivalence of (2), (3) and (4) follow from theorem (2.18).

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