Fibrewise Pairwise *bi*-Separation Axioms

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Abstract

The main idea of this research is to consider fibrewise pairwise versions of the more important separation axioms of ordinary bitopology named fibrewise pairwise bi-T_0 spaces, fibrewise pairwise bi -T_1 spaces, fibrewise pairwise bi-R_0 spaces, fibrewise pairwise bi-Hausdorff spaces, fibrewise pairwise functionally bi-Hausdorff spaces, fibrewise pairwise bi-regular spaces, fibrewise pairwise completely bi-regular spaces, fibrewise pairwise bi-normal spaces and fibrewise pairwise functionally bi-normal spaces. In addition we offer some results concerning it.

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1. Introduction

In order to begin the category in the classification of fibrewise (briefly F.W.) sets over a given set, named the base set, which say B. A F.W. set over B consest of a function $p: M \to B,$ that is named the projection on the set M. The fibre over b for every point b of B is the subset $M_b = p^{-1}(b)$ of M. Since we do not require p is surjectve, the fibre Perhaps, will be empty, also, for every B^* subset of B we considered $M_{B^*} = p^{-1}(B^*)$ like a F.W. set with the projection determined by p over B^*.

The another notation $M \mid B^*$ is some time fitting. We considered for every set T, the Cartesian product $B \times T$, by the first projection like a F.W. set B.

Definition 1.1, [4]:

If M and N with projections p_M and p_N, respectively, are F.W. sets over B, a function $\varphi: M \to N$ is named F.W. function if $p_N \text{ o } \varphi = p_M$, or $\varphi(M_b) \subset N_b$ for every $b \in B$.

Observe that a F.W. function $\varphi: M \to N$ over B limited by restriction, a F.W. function $\varphi_{B^*}: M_{B^*} \to N_{B^*}$ over B^* for every subset B^* of B.

Definition 1.2, [4]:

Let (B, Λ) be a topological space. The F.W. topology on a F.W. set M over B mean any topology on M makes the projection p is continuous.

Remark 1.3, [4]:

(a) The coarsest such topology is the topology made by p, in which the open sets of M are exactly the inverse image of the open sets of B; this is named F.W. indiscrete topology.

(b) The F.W. topological space over B is defined to be a F.W. set over B with a F.W. topology.

We consider the topology product $B \times_B T$, for every topological space T, like a F.W. topological spaces over B by the first projection. The equivalences in the type of F.W. topological spaces are named F.W. topological equivalences. We say that M is trivial, as a F.W. topological spaces over B, if for some topological space T, M is F.W. topologically equivalent to $B \times_B T$. In F.W. topology the word neighborhood (briefly nbd) is used in exactly in the similar sense like it is in ordinary topology, but the words F.W. basic may want some details, hence let M be F.W. topological space over B, if x is a point of M_b; b $\in B$, describe a family $N(x)$ of nbds of x in M as F.W. basic if for each nbd U of x where $M_W \cap V \subset U$, for a few member V of $N(x)$ and nbd W of b in B. For example, as in the topological product $B \times T$, where T is a topological space, the family of Cartesian products $B \times N(t)$, where $N(t)$ runs through the nbds of t, is F.W. basic for (b,t). Otherwise we follow closely James [4], Engelking [3] and Bourbaki [2].

Definition 1.4, [4]:

The F.W. function $\varphi: M \to N$, where M and N are F.W. topological spaces over B is named:

(a) Continuous if for every $x \in M_B$; $b \in B$, the inverse image of every open set of $\varphi(x)$ is an open set of x.

(b) Open if for every $x \in M_B$; $b \in B$, the direct image of every open set of x is an open set of $\varphi(x)$.

Definition 1.5, [3]:

Assume we are given a topological space M, a family $\{\varphi_s\}_{s\in S}$ of continuous functions, and a family $\{ N_s \}_{s\in S}$ of topological spaces where $\varphi_s : M \to N_s$ the function transfer $x \in M$ to the point $\{\varphi_s \ (x)\} \in \prod_{s\in S} N_s$ is continuous, it is named the diagonal of the functions $\{\varphi_s\}_{s\in S}$ and is denoted by $\Delta_{s\in S}\varphi_s$ or $\Delta\, \varphi_1 \Delta\, \varphi_2\, \Delta \ldots \Delta\, \varphi_k$ if S = {1, 2,..., k}.

Definition 1.6, [4]:

The F.W. topological space (M,τ) over (B,Λ) is named F.W. closed, (resp. F.W. open) if the projection p is closed (resp. open).

The bitopological spaces study was first created by Kelly [5] in 1963 and after that a large number of researches have been completed to generalize the topological ideas to bitopological setting. In this research (M,τ_1,τ_2) and (N,σ_1,σ_2) (or briefly, M and N) always mean bitopological spaces on which no separation axioms are supposed unless clearly stated. By τ_i-open (resp., τ_i-closed), we shall mean the open (resp., closed) set with respect to τ_i in M, where $i = 1,2$. A is open (resp., closed) if it is both τ_1-open (resp., τ_1-closed), τ_2-open (resp., τ_2-closed) in M. As well as, we built on some of the result in [1, 6, 9, 10, 11]. Otherwise we go behind closely I. M. James [4], R. Engelking [3] and N. Bourbaki [2].

Definition 1.7, [5]:

The triple (M,τ_1,τ_2) where M is a non-empty set and τ_1 and τ_2 are topologies on M is named bitopological spaces.

Definition 1.8, [5]:

A function $\varphi: (M,\tau_1,\tau_2) \to (N,\sigma_1,\sigma_2)$ is said to be τ_i-continuous (resp. τ_i-open and τ_i-closed), if the functions $\varphi: (M,\tau_i) \to (N,\sigma_i)$ are continuous (resp. open and closed), φ is named continuous (resp. open and closed) if it is τ_i-continuous (resp. τ_i-open and τ_i-closed) for every $i = 1,2$.

Definition 1.9, [8]:

Let (B,Λ_1,Λ_2) be a bitopological space. The F.W. bitopology on a F.W. set M over B mean any bitopology on M makes the projection p is continuous.

Definition 1.10, [7]:

A bitopological space (M,τ_1,τ_2) is said to be pairwise T_0 space if for every pair of points x and y such that $x \neq y$ there exists a τ_i-open set containing x but not containing y or a τ_j-open set containing y but not containing x, where $i,j = 1, 2, i \neq j$.

2. Fibrewise pairwise bi-T_0, pairwise bi-T_1, pairwise bi-R_0, and pairwise bi-Hausdorff spaces.

The concepts of pairwise open sets have an important role in F.W. separation axioms. By using these concepts we can construct many several F.W. separation axioms. Now we introduce the versions of F.W. pairwise bi-T_0, F.W. pairwise bi-T_1, F.W. pairwise bi-R_0, and F.W. pairwise bi-Hausdorff spaces as follows.

Definition 2.1:

Let (M,τ_1,τ_2) be F.W. bitopological space over (B,Λ_1,Λ_2). Then M is named F.W. pairwise bi-T_0 if whenever $x, y \in M_b$; $b \in B$ and $x \neq y$, either there exists a τ_i-open set U of x which does not contain y in M or τ_j-open set V of y which does not contain x in M, where $i,j = 1, 2, i \neq j$.

Remark 2.2:

(a) (M,τ_1,τ_2) is F.W. pairwise bi-T_0 space iff each fiber M_b is pairwise bi-T_0 space.

(b) Subspaces of F.W. pairwise bi-T_0 spaces are F.W. pairwise bi-T_0 spaces.

(c) The F.W. bitopological products of F.W. pairwise bi-T_0 spaces with the family of F.W. pairwise projections are F.W. pairwise bi-T_0 spaces.

For sure anyone can makes a F.W. version of the pairwise bi-T_1 space in a similar way. Let (M, τ_1, τ_2) be F.W. bitopological space over $(B, \Lambda_1, \Lambda_2)$. Then M is named F.W. pairwise *bi-T1* if whenever $x,\ y \in M_b;\ b\ \in B$ and $x \neq y$, there exist a τ_i -open sets U, and a τ_j -open set V in M such that $x \in U$, $y \notin U$ and $x \notin V, y \in V, i, j = 1, 2, i \neq j$. But it turns out that there is no real use for this in what we are going to do. In its place we formulate some use of a new axiom "The axiom is that every τ_i- open set contains the τ_j-closure of each of its points", and use the word pairwise bi-R_0 space. This is correct for pairwise bi-T_1 spaces and for pairwise bi-regular spaces. Thinking of it like a weak structure of pairwise bi-regularity. For example, indiscrete spaces are pairwise bi-R_0 spaces. The F.W. version of the pairwise bi-R_0 axiom as the following.

Definition 2.3:

A F.W. bitopological space (M, τ_1, τ_2) over $(B, \Lambda_1, \Lambda_2)$ is named F.W. pairwise bi-R_0 if for every $x \in M_b$; $b \in B$, and every τ_i-open set V of x in M, there exists a nbd W of b in B where V is containing the τ_j- closure of $\{x\}$ in M_W is (i.e., $M_W\ \cap \tau_j$- $Cl\{x\}\ \subset\ V$) where $i, j = 1, 2,\ i \neq j$.

For example, $(B, \Lambda_1, \Lambda_2) \times_B (T, \tau_1, \tau_2)$ is F.W. pairwise bi-R_0 space for all pairwise bi-R_0 spaces T.

Remark 2.4:

(a) The nbds of x are given by a F.W. basis it is enough if the condition in Definition (2.3) is satisfied for every F.W. basic nbds.

(b) If (M, τ_1, τ_2) is F.W. pairwise bi-R_0 space over $(B, \Lambda_1, \Lambda_2)$, then for each subspace $(B^*, \Lambda_1^*, \Lambda_2^*)$ of $(B, \Lambda_1, \Lambda_2)$ $(M_B^*, \tau_1^*, \tau_2^*)$ is F.W. pairwise bi-R_0 space over B^*.

Proposition 2.5:

Let $\varphi\ : M \to M^*$ be a continuous F.W. embedding function, where (M, τ_1, τ_2) and $(M^*, \tau_1^*, \tau_2^*)$ are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. If M^* is F.W. pairwise bi-R_0 then so is M.

Proof:

Let V be a τ_i-open set of x in M, where $x \in M_b$; b $\in B$. Then $V = \varphi^{-1}(V^*)$, where V^* is a ${\tau^*}_i$-open set of $x^* = \varphi(x)$ in M^*. Because M^* is F.W. pairwise bi-R_0 then we have a nbd. W of b in B, where ${M^*}_W \cap \tau_j^*$-$Cl\{x^*\}\ \subset\ V^*$. Hence, $M_W\ \cap\ \tau_j$ -$Cl\{x\} \subset\ \varphi^{-1}({M^*}_W \cap\ \tau_j^*$-$Cl\{x^*\}) \subset \varphi^{-1}(V^*) = V$ and hence M is F.W. pairwise bi-R_0 where $i, j = 1, 2,, i \neq j$.

The class of F.W. pairwise bi_R_0 spaces is finitely multiplicative, like in the following.

Proposition 2.6:

If $\{(M_r, \tau_{1r}, \tau_{2r})\}$ is a finite family of F.W. pairwise bi-R_0 spaces over B. Then the F.W. bitopological product $M = \prod_B M_r$ is F.W. pairwise bi-R_0.

Proof:

Let $x \in M_b;\ b \in B$. Consider a τ_i-open set $V = \prod_B V_r$ of x in M, where V_r is a τ_{ir}-open set of $\pi_r(x) = x_r$ in M_r for each index r. Since M_r is F.W. pairwise bi-R_0 then, we have a nbd W_r of b in B where $(M_r \mid W_r) \cap\ \tau_{jr}$-$Cl\{x_r\}$ $\subset V_r$. Then we regard W as a nbd of b where W is an intersection of W_r and $M_W \cap\ \tau_j$-$Cl\{x\}$ $\subset V$ and hence $M = \prod_B M_r$ is F.W. pairwise bi-R_0 where i, j =1,2,, $i \neq j$.

The similar conclusion holds for infinite F.W. products provided all of the factors is F.W. nonempty.

Proposition 2.7:

Assume that $\varphi\colon M \to N$ is closed, continuous F.W. surjection function, where (M, τ_1, τ_2) and (N, σ_1, σ_2) are F.W. bitopological spaces over B. If M is F.W. pairwise bi-R_0 then so is N.

Proof:

Assume that V is a σ_i-open set of y in N, where $y\ \in N_b$; $b \in B$, choose $x \in \varphi^{-1}(y)$. Then $U\ =\ \varphi^{-1}(V)$ is a τ_i-open set of x in M. because M is F.W. pairwise bi-R_0, then we have a nbd W of b in B, where $M_W\ \cap \tau_j - Cl\{x\}\ \subset\ U$. Therefore $N_W\ \cap\ \varphi(\tau_j - Cl\{x\})\ \subset \varphi(U)\ =\ V$. because $\varphi$is closed, $\varphi\big(\tau_j - Cl\{x\}\big) =\ \sigma_j - Cl(\varphi\{x\})$. Hence $NW\ \cap\ \sigma_j - Cl(\varphi\{x\}) \subset\ V$ and N is F.W. pairwise bi-R_0 where $i, j = 1, 2, i \neq j$.

Now we introduce the version of F.W. pairwise bi-Hausdorff spaces like the following.

Definition 2.8:

A F.W. bitopological space (M, τ_1, τ_2) over $(B, \Lambda_1, \Lambda_2)$ is named F.W. pairwise bi-

Hausdorff if whenever $x, y \in M_b$; $b \in B$ and $x \neq y$ there exist a disjoint pair of τ_i-open set U of x and τ_j-open set V of y in M, where $i, j = 1, 2, i \neq j$.

For example, $(B, \Lambda_1, \Lambda_2) \times_B (T, \tau_1, \tau_2)$ is F.W. pairwise bi-Hausdorff space for all pairwise bi-Hausdorff spaces T.

Remark 2.9:

If (M, τ_1, τ_2) is F.W. pairwise bi-Hausdorff space over $(B, \Lambda_1, \Lambda_2)$ then M_B^* is F.W. pairwise bi-Hausdorff over B^* for every subspace B^* of B. Specially the fibers of (M, τ_1, τ_2) are pairwise bi-Hausdorff spaces. On the other hand a F.W. bitopological space with pairwise bi-Hausdorff fibres is not necessarily pairwise bi-Hausdorff.

Example:2.10:

Let M={1, 2, 3}, $\tau_1 = \{M, \varphi, \{1\}, \{1,2\}\}, \tau_2 = \{M, \varphi, \{1\}, \{1,3\}\}$. Let B={a, b}, $\Lambda_1 = \{B, \varphi, \{a\}\}, \Lambda_2 = I$. Let p: M→ B where: p (1) = a, p (2) = b = p (3). Then, we have $M_b = \{2,3\}$, $\tau_{1Mb} = \{M_b, \varphi, \{2\}\}, \tau_{2Mb} = \{M_b, \varphi, \{3\}\}$. Then $\exists\ \tau_{1Mb}$-open set U = {2} where $2 \in U, 3 \notin U$ and there exist τ_{2Mb} open set V = {3} where $3 \in V, 2 \notin V$, where $U \cap V = \varphi$. But M is not pairwise bi-Hausdorff since: 2 and $3 \in M$ and $2 \neq 3$, and there is no disjoint pair of open sets of 2 and 3.

Proposition 2.11:

The F.W. bitopological space (M, τ_1, τ_2) over $(B, \Lambda_1, \Lambda_2)$ is F.W. pairwise bi-Hausdorff iff the diagonal embedding Δ: $M \to M \times_B M$ is $\tau_i \times_B \tau_i$ -closed.

Proof:

(⇒) Let $x, y \in Mb$; b ∈ B and $x \neq y$. Since $\Delta(M)$ is $\tau_i \times_B \tau_i$-closed in $M \times_B M$, then (x, y) a point of the complement, admits a F.W. product $\tau_i \times_B \tau_j$-open set $U \times_B V$ which does not meet $\Delta(M)$, and then U, V are disjoint pair of x, y. Where U is τ_i-open set of x, and V is τ_j-open set of y, where $i, j = 1, 2, i \neq j$.

(⇐) The reverse direction is similar.

Subspaces of F.W. pairwise bi-Hausdorff spaces are F.W. pairwise bi-Hausdorff spaces. Actuality we have.

Proposition 2.12:

Assume that $\varphi : M \to M^*$ is a continuous embedding F.W. function, where (M, τ_1, τ_2) and $(M^*, \tau^*{}_1, \tau^*{}_2)$ are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. If M^* is F.W. pairwise bi-Hausdorff then so is M.

Proof:

Let $x, y \in Mb$; $b \in B$ and $x \neq y$. Then $\varphi(x), \varphi(y) \in {M_b}^*$ are distinct, since M^* is F.W. pairwise bi-Hausdorff, then we have a τ_i^*-open sets U^* of $\varphi(x)$ and τ_j^*-open set V^* of $\varphi(y)$ in M^* which are disjoint. Because φ is continuous, the inverse images $\varphi^{-1}(U^*) = U$, $\varphi^{-1}(V^*) = V$, such that U is τ_i-open set of x and V is τ_j-open set of y in M which are disjoint and so M is F.W. pairwise bi-Hausdorff where $i, j = 1, 2, i \neq j$.

Proposition 2.13:

Let $\varphi : M \to N$ be a continuous F.W. function, where (M, τ_1, τ_2) and (N, σ_1, σ_2) are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. If N is F.W. pairwise bi-Hausdorff then the F.W. graph $\Gamma : M \to M \times B\ N$ of φ is a $\tau_i \times_B \sigma_j$-closed embedding.

Proof:

The F.W. graph is defined in the similar method like the ordinary graph, but with values in the F.W. product, hence the figure shown below is commutative.

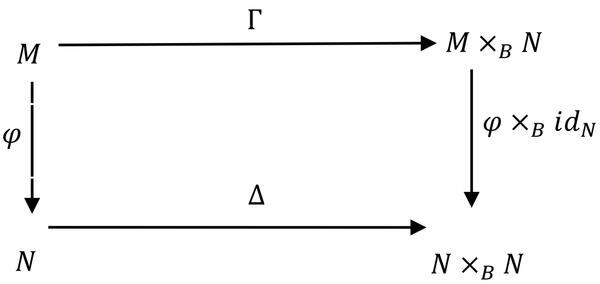

Fig. (1.1) Diagram of Proposition 2.13.

Since $\Delta(N)$ is $\sigma_i \times_B \sigma_i$-closed in $N \times BN$, by Proposition (2.11), so $\Gamma(M) = (\varphi \times idN)^{-1}(\Delta(N))$ is $\tau_i \times_B \sigma_j$-closed in $M \times_B N$, as asserted, where $i, j = 1, 2, i \neq j$.

The category of F.W. pairwise bi-Hausdorff spaces is multiplicative, like the following sense.

Proposition 2.14:

Assume that $\{(Mr, \tau_{1r}, \tau_{2r})\}$ is a family of F.W. pairwise bi-Hausdorff spaces over

$(B, \Lambda_1, \Lambda_2)$. The F.W. bitopological product $M = \prod_B M_r$ with the family of F.W. projection $\pi r : M = \prod_B M_r\, M \rightarrow M_r$ is F.W. pairwise bi-Hausdorff.

Proof:

Let $x, y \in Mb$; $b \in B$ and $x \neq y$. Then $\pi_r(x) = x_r \neq \pi_r(y) = y_r$ for some index *r*. Because M_r is F.W. pairwise bi-Hausdorff, then we have a τ_{ir}-open set U_r of x_r, and τ_{jr}-open set V_r of y_r in M_r which are disjoint. Because π_r is continuous, the inverse images U, V are disjoint τ_i-open and τ_j-open sets respectively of x, y in M, where $i, j = 1, 2, i \neq j$.

The pairwise functionally version of the F.W. pairwise bi-Hausdorff axiom is stronger than the non pairwise functional version but its properties are quite like. Here and in another place we use I to mean the closed unit interval [*0*, *1*] in the real line ℝ.

Definition 2.15:

A F.W. bitopological space (M, τ_1, τ_2) over $(B, \Lambda_1, \Lambda_2)$ is F.W. pairwise functionally bi-Hausdorff if for every $x, y \in M_b$; b ∈ B and $x \neq y$, there exists a nbd *W* of *b* in *B* and disjoint pair τ_i-open sets U of x and τ_j-open set V of y in M and a continuous function $\lambda: M_W \rightarrow I$ where $M_b \cap U \subset \lambda^{-1}(0)$ and $M_b \cap V \subset \lambda^{-1}(1)$ where $i, j = 1, 2, i \neq j$.

For example, $(B, \Lambda_1, \Lambda_2) \times_B (T, \tau_1, \tau_2)$ is F.W. pairwise functionally bi-Hausdorff space for each pairwise functionally bi-Hausdorff spaces *T*.

Remark 2.16:

If (M, τ_1, τ_2) is F.W. pairwise functionally bi-Hausdorff space over $(B, \Lambda_1, \Lambda_2)$ then M_B^* is F.W. pairwise functionally bi-Hausdorff over B^* for every subspace B^* of *B*. In particular the fibres of *M* are pairwise functionally bi-Hausdorff spaces.

Subspaces of F.W. pairwise functionally bi-Hausdorff spaces are F.W. pairwise functionally bi-Hausdorff spaces. Actuality we have.

Proposition: 2.17.

Assume that $\varphi : M \rightarrow M^*$ is a continuous embedding F.W. function, where (M, τ_1, τ_2) and $(M^*, \tau_1^*, \tau_2^*)$ are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. If M^* is F.W. pairwise functionally bi-Hausdorff then so is *M*.

Proof:

Let $x, y \in M_b$ and $x \neq y; b \in B$. Then $\varphi(x) = x^*, \varphi(y) = y^* \in M_b^*$, $x^* \neq y^*$. Since M^* is F.W. pairwise functionally bi-Hausdorff, then we have a nbd W of b in B and disjoint pair of τ_i^*-open set U^* of x^* and τ_j^*-open set V^* of y^* and a continuous function $\lambda^*: M^* \mid W \rightarrow I$ where $M_b^* \cap U^* \subset (\lambda^*)^{-1}(0)$ and $M_b^* \cap V^* \subset (\lambda^*)^{-1}(1)$. Now, since φ is continuous, $\varphi^{-1}(U^*) = U$ and $\varphi^{-1}(V^*) = V$ are disjoint pair of τ_i-open set of x and τ_j-open set of y respectively and the continuous function λ where $\lambda = \lambda^* o\, \varphi: M_W \rightarrow I$ such that $M_b \cap U \subset \lambda^{-1}(0)$ and $M_b \cap V \subset \lambda^{-1}(1)$, where $i, j = 1, 2, i \neq j$.

Furthermore the category of F.W. pairwise functionally bi-Hausdorff spaces is multiplicative, as the following.

Proposition 2.18:

Assume that $\{(M_r, \tau_{1r}, \tau_{2r})\}$ is a family of F.W. pairwise functionally bi-Hausdorff spaces over $(B, \Lambda_1, \Lambda_2)$. The F.W. bitopological product $M = \prod_B M_r$ with the family of F.W. projection $\pi r : M = \prod_B M_r \rightarrow M_r$ is F.W. pairwise functionally bi-Hausdorff.

Proof:

Let $x, y \in M_b; b \in B$, and $x \neq y$. Then $\pi_r(x) = x_r, \pi_r(y) = y_r \in (M_r)_b$ for some index r where $x_r \neq y_r$. Since Mr is F.W. pairwise functionally bi-Hausdorff, then, we have a nbd W_r of b in B and disjoint pair of τ_{ir}-open set U_r of x_r, and τ_{jr}-open set V_r of y_r and a continuous function $\lambda: M_r \mid W_r \rightarrow I$ such that $(M_r)_b \cap U_r \subset \lambda^{-1}(0)$ and $(M_r)_b \cap V_r \subset \lambda^{-1}(1)$. Now the intersection of W_r is a nbd W of b in B, and since π_r is continuous, then $\pi_r^{-1}(U_r) = U$ and $\pi_r^{-1}(V_r) = V$ are disjoint pair of τ_i-open set of x and τ_j-open set of y respectively and the continuous function Ω where $\Omega = \lambda o \pi_r: M_W \rightarrow I$ where $M_b \cap U \subset \Omega^{-1}(0)$ and $M_b \cap V \subset \Omega^{-1}(1)$ where $i, j = 1, 2, i \neq j$.

3. Fibrewise pairwise bi-regular and pairwise bi-normal spaces

We at this time go on to consider the F.W. versions of the advanced pairwise separation axioms, first with F.W. pairwise bi-regularity and F.W. pairwise completely bi-regularity.

Definition: 3.1.

The F.W. bitopological space (M, τ_1, τ_2) over $(B, \Lambda_1, \Lambda_2)$ is named F.W. pairwise bi-regular if for every $x \in M_b$; $b \in B$, and for every τ_i-open set V of x in M, there exists a nbd. W of b in B, and a τ_i- pen set U of x in MW such that V is containing the τ_j-closure of U in MW $\left(i.e., M_W \cap \tau_j - Cl(U) \subset V\right)$, where $i, j = 1, 2, i \neq j$.

For example, trivial F.W. spaces with pairwise bi-regular fibre are F.W. pairwise bi-regular.

Remark 3.2:

(a) The nbds of x are given by a F.W. basis it is enough if the condition in Definition (3.1) is satisfied for every F.W. basic nbds.

(b) If (M, τ_1, τ_2) is F.W. pairwise bi-regular space over $(B, \Lambda_1, \Lambda_2)$ then $(M_B^*, \tau_1^*, \tau_2^*)$ is F.W. pairwise bi-regular space over $(B^*, \Lambda_1^*, \Lambda_2^*)$ for every subspace B^* of B.

Subspaces of F.W. pairwise bi-regular spaces are F.W. pairwise bi-regular spaces. Actuality we have.

Proposition 3.3:

Assume that $\varphi : M \rightarrow M^*$ is a continuous embedding F.W. function, where (M, τ_1, τ_2) and $(M^*, \tau^*{}_1, \tau^*{}_2)$ are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. If M^* is F.W. pairwise bi-regular then so is M.

Proof:

Let V be a τ_i-open set of x in M, where $x \in M_b$; $b \in B$. Then $V = \varphi^{-1}(V^*)$, where V^* is a τ_i^*-open set of $x^* = \varphi(x)$ in M_b^*. Because M* is F.W. pairwise bi-regular then, we have a nbd W of b in B and a τ_i^*-open set U^* of x* in M_W^* where $M_W^* \cap \tau_j^* - cl(U^*) \subset V^*$. Then $U = \varphi^{-1}(U^*)$ is a τ_i-open set of x in M_W such that $M_W \cap \tau_j - Cl(U) \subset V$, hence M is F.W. pairwise bi-regular, where $i, j = 1, 2,\ i \neq j$ as required.

The class of F.W. pairwise bi-regular spaces is F.W. multiplicative, like in the following.

Proposition 3.4:

Assume that $\{(M_r, \tau_{1r}, \tau_{2r})\}$ is a finite family of F.W. pairwise bi-regular spaces over B. The F.W. bitopological product $M = \prod_B M_r$ is F.W. pairwise bi-regular.

Proof:

Consider a τ_i-open set $V = \prod_B V_r$ of x in M, where $x \in M_b$; b ∈ B and V_r is a τ_{ir}-open set of $\pi_r(x) = x_r$ in M_r for each index r. Since M_r is F.W. pairwise bi-regular we have a nbd. W_r of b in B, and a τ_{ir}-open set U_r of x_r in $M_r \mid W_r$ such that the τ_{jr}-closure of U_r in $M_r \mid W_r$ is contained in V_r. (i. e. $(M_r \mid W_r) \cap \tau_{jr} - Cl(U_r) \subset V_r$). Then we regard Was a nbd of b in B, where W is the intersection of W_r, and $U = \prod_B U_r$ is a τ_i-open set of x in M_W where the τ_j-closure of U in M_W is contained in V. (i. e. $M_W \cap \tau_j - cl(U) \subset V$), and so $M = \prod_B M_r$ is F.W. pairwise bi-regular, where $i, j = 1, 2,\ i \neq j$.

The similar conclusion holds for infinite F.W. products provided every of the factors is F.W. non-empty.

Proposition 3.5:

Assume that $\varphi : M \rightarrow N$ is a closed, open and continuous F.W. surjection function, where (M, τ_1, τ_2) and (N, σ_1, σ_2) are F.W. bitopological spaces over B. Then M is F.W. pairwise bi-regular iff N is F.W. pairwise bi-regular.

Proof:

(⇒) Let V be a σ_i-open set of y in N where $y \in N_b$; $b \in B$, choose $x \in \varphi^{-1}(y)$. Then $U = \varphi^{-1}(V)$ is a τ_i-open set of x in M. Because M is F.W. pairwise bi-regular, we have a nbd W of b in B, and a τ_i- open set U^* of x such that $M_W \cap \tau_j - cl(U^*) \subset U$. Then $N_W \cap \varphi\left(\tau_j - cl(U^*)\right) \subset V$. Because φ is closed, $\varphi\left(\tau_j - cl(U^*)\right) = \sigma_j - cl(\varphi(U^*))$ and because φ is open, then $\varphi(U^*)$ is a σ_i-open set of y. Hence N is F.W. pairwise bi-regular, where $i, j = 1, 2,\ i \neq j$, as asserted.

(⇐) By similar way of first direction.

The pairwise functionally version of the F.W. pairwise bi-regularity axiom is stronger than the non-pairwise functionally version but its properties are quite like. In the ordinary theory the word completely bi-regular is all the time used instead of functionally bi-regular and we widen this usage to the F.W. theory.

Definition 3.6:

A F.W. bitopological space (M,τ_1,τ_2) over (B,Λ_1,Λ_2) is named F.W. pairwise completely bi-regular if for every $x\in M_b$; $b\in B$, and for every τ_i-open set V of x there exists a nbd W of b in B and a τ_j-open set U of x in M_W and a continuous function $\lambda:(M_W,\tau_{1W},\tau_{2W})\to I$ such that $M_b\cap U\subset\lambda^{-1}(0)$ and $M_W\cap(M_W-V)\subset\lambda^{-1}(1)$, where $i,j=1,2, i\neq j$.

For example, $(B,\Lambda_1,\Lambda_2)\times_B(T,\tau_1,\tau_2)$ is F.W. pairwise completely bi-regular space for every pairwise completely bi-regular spaces T.

Remark: 3.7.

(a) The nbds. of x are given by a F.W. basis it is enough if the condition in Definition (3.6) is satisfied for every F.W. basic nbds.

(b) If (M,τ_1,τ_2) is F.W. pairwise completely bi-regular space over (B,Λ_1,Λ_2) then $(M_B^*,\tau_1^*,\tau_2^*)$ is F.W. Pairwise completely bi-regular space over (B*, Λ_1^*,Λ_2^*) for every subspace B* of B.

Subspaces of F.W. pairwise completely bi-regular spaces are F.W. pairwise completely bi-regular spaces. In fact we have.

Proposition 3.8:

Assume that $\varphi: M\to M^*$ is a continuous embedding F.W. function, where (M,τ_1,τ_2) and (M^*,τ_1^*,τ_2^*) are F.W. bitopological spaces over (B,Λ_1,Λ_2). If M* is F.W. pairwise completely bi-regular then so is M.

Proof:

Let V be a τ_i-open set of x in M where $x\in M_b; b\in B$, then $\varphi(x)=x^*\in M_b^*$ and $V=\varphi^{-1}(V^*)$ is a τ_i^*-open set of x^*. Because M^* is F.W. pairwise completely bi-regular, then we have a nbd. W of b in B and τ_j^*-open set U^* of x^* and a continuous function $\lambda: M_W^*\to I$ such that $M_b^*\cap U^*\subset\lambda^{-1}(0)$ and $M_W^*\cap(M_W^*-V^*)\subset\lambda^{-1}(1)$. Now, because φ is continuous, then $\varphi^{-1}(U^*)=U$ is τ_i-open set of x in M_W and the continuous function $\Omega=\lambda o\varphi$ such that $\Omega: M_W\to I$ and $M_b\cap U\subset\Omega^{-1}(0)$ and $M_W\cap(M_W-V)\subset\Omega^{-1}(1)$, where $i,j=1,2, i\neq j$.

The class of F.W. pairwise completely bi-regular spaces is finitely multiplicative, like in the following.

Proposition 3.9:

Assume that $\{(M_r,\tau_{1r},\tau_{2r})\}$ is a finite family of F.W. pairwise completely bi-regular spaces over (B,Λ_1,Λ_2). The F.W. bitopological product $M=\prod_B M_r$ is F.W. pairwise completely bi-regular.

Proof:

Let $x\in M_b$; $b\in B$. Consider a F.W. τ_i-open set $\prod_B V_r$ of x in M, where V_r is a τ_{ir}-open set of $\pi_r(x)=x_r$ in M_r for all index r. Because M_r is F.W. pairwise completely bi-regular, we have a nbd. W_r of b *in* B, and a τ_{jr}-open set U of x_r in M_r and a continuous function $\lambda_r:(M_r)_W\to I$ where $(M_r)_b\cap U\subset\lambda_r^{-1}(0)$ and $(M_r)_W\cap((M_r)_W-V_r)\subset\lambda_r^{-1}(1)$. Then we regard W like a nbd of b in B where W is the intersection of W_r and $\lambda: M_W\to I$ is a continuous function where:

$\lambda(\xi)=inf_{r=1,2,3,\dots,n}\{\lambda_r\xi_r\}$ for $\xi=(\xi_r)\in M_W$.

Since:

$$(M_r)_b\cap\pi_r{}^{-1}(U)\subset\pi_r{}^{-1}[(M_r)_b\cap U]$$
$$\subset\pi_r{}^{-1}\left(\lambda^{-1}{}_r(0)\right)$$
$$=(\lambda_r o\pi_r)^{-1}(0)$$

and

$$(Mr)_W\cap\pi_r{}^{-1}((M_r)_w-V_r)\subset\pi_r{}^{-1}[(M_r)_W\cap((M_r)_W-V_r)]\subset\pi_r{}^{-1}\left(\lambda_r{}^{-1}(1)\right)$$
$$=(\lambda_r o\pi_r)^{-1}(1).$$

where $i,j=1,2, i\neq j$.

The similar conclusion holds for infinite F.W. products provided that all of the factors is F.W. non-empty.

Lemma 3.10:

Assume that $\varphi: M\to N$ is a closed, open F.W. surjection function, where M and N are F.W. bitopological spaces over B. Let $\alpha: M\to\mathbb{R}$ be a continuous real-valued function which is F.W. bounded above, in the sense that α is bounded above on each fibre of M. Then $\beta: N\to\mathbb{R}$ is continuous, where:

$\beta(\eta)=sup_{\xi\in\varphi^{-1}(\eta)}\alpha(\xi)$

Proposition 3.11:

Assume that $\varphi: M\to N$ is a closed, open and continuous F.W. surjection function, where (M,τ_1,τ_2) and (N,σ_1,σ_2) are F.W. bitopological spaces over (B,Λ_1,Λ_2). If M is F.W. pairwise completely bi-regular then so is N.

Proof:

Let V_y be a σ_i-open set of y in N, where $y \in N_b$; $b \in B$. Choose $x \in \varphi^{-1}(y)$, so that $V_x = \varphi^{-1}(V_y)$ is a τ_i-open set of x. Because M is F.W. pairwise completely bi-regular, we have a nbd. W of b in B, and a τ_j-open set U_x of x in M_W and a continuous function $\lambda: M_W \to I$ such that $M_b \cap U_x \subset \lambda^{-1}(0)$ and $M_W \cap (M_W - V_x) \subset \lambda^{-1}(1)$. Using Proposition lemma (3.10), we get a continuous function $\Omega: N_W \to I$ such that $N_b \cap U_y \subset \Omega^{-1}(0)$ and $N_W \cap (N_W - V_y) \subset \Omega^{-1}(1)$, where $i, j = 1, 2, i \neq j$.

Now we define the version of F.W. pairwise normal space like in the following.

Definition 3.12:

A F.W. bitopological space (M, τ_1, τ_2) over $(B, \Lambda_1, \Lambda_2)$ is named F.W. pairwise bi-normal if for every $b \in B$ and every disjoint pair of τ_i-closed set H, and τ_j-closed set K of M, there exists a nbd. W of b in B and a disjoint pair of τ_j-open set U, and τ_i-open set V of $M_W \cap H, M_W \cap K$ in MW, where $i, j = 1, 2, i \neq j$.

Remark 3.13:

If (M, τ_1, τ_2) is F.W. pairwise bi-normal space over $(B, \Lambda_1, \Lambda_2)$, then for each subspace B^* of B, $(M_B^*, \tau_1^*, \tau_2^*)$ is F.W. pairwise bi-normal space over $(B^*, \Lambda_1^*, \Lambda_2^*)$.

Closed subspaces of F.W. pairwise bi-normal spaces are F.W. pairwise bi-normal. Actuality we have.

Proposition 3.14:

Assume that $\varphi: M \to M^*$ is a closed, continuous embedding F.W. function where (M, τ_1, τ_2) and $(M^*, \tau_1^*, \tau_2^*)$ are F.W. bitopological spaces over B. If $(M^*, \tau_1^*, \tau_2^*)$ is F.W. pairwise bi-normal then so is (M, τ_1, τ_2).

Proof:

Let H, K be disjoint pair of τ_i-closed, and τ_j-closed sets of M and let b $\in B$. Then $\varphi(H), \varphi(K)$ are disjoint pair of ${\tau_i}^*$-closed set and ${\tau_j}^*$-closed set of M^*. Since M^* is F.W. pairwise bi-normal then, we have a nbd W of b in B and a ${\tau_j}^*$-open set U^* and ${\tau_i}^*$-open set V^* of $M_W^* \cap \varphi(H), M_W^* \cap \varphi(K)$, in M_W^*. $\varphi^{-1}(U^*) = U$ and $\varphi^{-1}(V^*) = V$ are disjoint pair of τ_j-open and τ_i-open sets of $M_W \cap H, M_W \cap K$ in M_W, where $i, j = 1, 2, \ i \neq j$.

Proposition: 3.15.

Let φ: $M \to N$ be a closed continuous F.W. surjection function, where (M, τ_1, τ_2) and (N, σ_1, σ_2) are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. Then (M, τ_1, τ_2) is F.W. pairwise bi-normal iff (N, σ_1, σ_2) is F.W. pairwise bi-normal.

Proof:

(⇒) Let H, K be disjoint pair of σ_i-closed, and σ_j-closed sets of N and let b $\in B$. $\varphi^{-1}(H), \varphi^{-1}(K)$ are disjoint pair of τ_i-closed and τ_j-closed sets of M. because M is F.W. pairwise bi-normal then, we have a nbd. W of b in B and a disjoint pair of τ_j-open set and τ_i-open set U, V of $M_W \cap \varphi^{-1}(H)$ and $M_W \cap \varphi^{-1}(K)$. Because φ is closed then, the sets $N_W - \varphi(M_W - U)$, $N_W - \varphi(M_W - V)$ are open in N_W, and structure a disjoint pair of σ_j-open, σ_i-open sets of $N_W \cap H$, $N_W \cap K$ in N_W, as required, where $i, j = 1, 2, i \neq j$.

(⇐) By similar way of first direction.

Lastly, we define the version of F.W. pairwise functionally bi-normal space like in the following.

Definition: 3.16.

A F.W. bitopological space (M, τ_1, τ_2) over $(B, \Lambda_1, \Lambda_2)$ is named F.W. pairwise functionally bi-normal if for every b ∈ B and every disjoint pair of τ_i-closed set H, and τ_j-closed set K of M, there exists a nbd. W of b in B and a disjoint pair of τ_j-open set U, and τ_i-open set V and a continuous function $\lambda: M_W \to I$ such that $M_W \cap H \cap U \subset \lambda^{-1}(0)$ and $M_W \cap K \cap V \subset \lambda^{-1}(1)$ in M_W, where $i, j = 1, 2, i \neq j$.

For example, $(B, \Lambda_1, \Lambda_2) \times_B (T, \tau_1, \tau_2)$ is F.W. pairwise functionally bi-normal space when T is pairwise functionally bi-normal space.

Remark 3.17:

If (M, τ_1, τ_2) is F.W. pairwise functionally bi-normal space over $(B, \Lambda_1, \Lambda_2)$ then for every subspace B^* of B we have $(M_B^*, \tau_1^*, \tau_2^*)$ is F.W. pairwise functionally bi-normal space over $(B^*, \Lambda_1^*, \Lambda_2^*)$.

Closed subspaces of F.W. pairwise functionally bi-normal spaces are F.W. pairwise functionally bi-normal. Actuality we have.

Proposition 3.18:

Assume that $\varphi: M \to M^*$ is a closed, continuous embedding F.W. function where (M, τ_1, τ_2) and $(M^*, \tau_1^*, \tau_2^*)$ are F.W. bitopological spaces over B. If M^* is F.W. pairwise functionally bi-normal then M is so.

Proof:

Let H, K be disjoint pair of τ_i-closed and τ_j-closed sets of M and let $b \in B$. Then $\varphi(H), \varphi(K)$ are disjoint pair of ${\tau_i}^*$-closed set and ${\tau_j}^*$- closed set of M^*. Since M^* is F.W. pairwise functionally bi-normal we have a nbd W of b in B and a disjoint pair of τ_j^*-open set U and τ_i^*-open set V and a continuous function $\lambda : M_W^* \to I$ such that $M_W^* \cap \varphi(H) \cap U \subset \lambda^{-1}(0)$ and $M_W^* \cap \varphi(K) \cap V \subset \lambda^{-1}(1)$ in M_W^*. Since φ is continuous, then $\varphi^{-1}(U), \varphi^{-1}(V)$ are τ_j-open set, τ_i-open set and the function, $\Omega = \lambda \, o \, \varphi$ is a continuous, $\Omega: M_W \to I$ such that $M_W \cap H \cap \varphi^{-1}(U) \subset \Omega^{-1}(0)$ and $M_W \cap K \cap \varphi^{-1}(V) \subset \Omega^{-1}(1)$ in M_W as required where $i, j = 1, 2, i \neq j$.

Proposition 3.19:

Assume that $\varphi: M \to N$ is a closed, open and continuous F.W. surjection function, where (M, τ_1, τ_2) and (N, σ_1, σ_2) are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. If (M, τ_1, τ_2) is F.W. pairwise functionally bi-normal then so is (N, σ_1, σ_2).

Proof:

Let H, K be disjoint pair of σ_i-closed, and σ_j-closed sets of N and let b ∈ B. Then $\varphi^{-1}(H), \varphi^{-1}(K)$ are disjoint pair of τ_i-closed and τ_j-closed sets of M. Because M is F.W. pairwise functionally bi-normal, then we have a nbd W of b in B and a disjoint pair of τ_j-open set and τ_i-open set U, V and a continuous function $\lambda: M_W \to I$ such that $M_W \cap \varphi^{-1}(H) \cap U \subset \lambda^{-1}(0)$ and $M_W \cap \varphi^{-1}(K) \cap V \subset \lambda^{-1}(1)$ in M_W. Hence a function $\Omega: N_W \to I$ is given by $\Omega(y) = \sup_{x \in \varphi^{-1}(y)} \lambda(x); y \in N_W$.

Because φ is open and closed, in addition to continuous, it leads to that Ω is continuous. Hence $N_W \cap H \cap \varphi(U) \subset \Omega^{-1}(0)$ and $N_W \cap K \cap \varphi(V) \subset \Omega^{-1}(1)$ in M_W where $i, j = 1, 2, i \neq j$.

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