Rationally Extending Modules and Strongly Quasi-Monoform Modules

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الخلاصة

بأنه مقاس توسيع برشد، إذا كان كل مقاس جزئي منه R المعرف على الحلقةMيقال للمقاس
. في هذا البحث ندرس هذا الصنف من المقاسات الذي يكون Mراشدا" في مركبة مجموع مباشر للمقاس
محتويا" في صنف مقاسات التوسيع. كذلك تأملنا صنف مقاسات شبه أحادية الصيغة بقوة، حيث يقال للمقاس
أنه شبه أحادي الصيغة بقوة، إذا كان كل مقاس غير صفري وفعلي فيه يكون شبه عكوسا" بالنسبة الى M
. إقترحنا شروطا توحد بين هذه الاصناف من المقاسات. جملة من الخواص Mمركبة مجموع مباشر للمقاس
أعطيت لهذا الصنف من المقاسات.

ABSTRUCT

An R-module M is called rationally extending if each submodule of M is rational in a direct summand of M. In this paper we study this class of modules which is contained in the class of extending modules, Also we consider the class of strongly quasi-monoform modules, an R-module M is called strongly quasi-monoform if every nonzero proper submodule of M is quasi-invertible relative to some direct summand of M. Conditions are investigated to identify between these classes. Several properties are considered for such modules.

INTRODUCTION

Throughout this paper R represents an associative ring with identity and all R-modules are unital right modules. The following are equivalent of a submodule N of an R-module M:(1) $Hom_R(M/N,E(M))=0$ where E(M) is the injective envelope of M, (2) For each submodule K of M with $N\subseteq K\subseteq M$, every R-homomorphism $\varphi: K\to M$ with $N\subseteq \ker(\varphi)$ is trivial and (3) For each x , $y\in M$ with $x \neq 0$, $x[N:y] \neq 0$ where $[N:y]=\{r\in R: ry\in N \}$. A submodule N of an R-module is called rational in M if N satisfies any one of the above conditions [1],[2].It is clear that rational submodules are refinement of essential submodules. An R-module M is called monoform (some times termed strongly uniform) if each non-zero submodule N of M is rational [3].A submodule N of an R-module M is called quasi-invertible if $Hom_R(M/N, M)=0$, and the quasi-dedekind are those in which all non-zero submodules are quasi-invertible [4]. Clearly, every rational submodule is quasi-invertible and hence every monoform R-module is quasi-dedekind.

The authors in [5] introduced relative quasi-invertible submodules. A proper submodule N of an R-module N is called quasi-invertible relative to a submodule P of M if P contains N properly and $Hom_R(P/N, M)=0$. An R-module M is called is called quasi-monoform if each non-zero proper submodule of M is quasi-invertible relative to some submodule of M. In fact quasi-invertible submodules are quasi-invertible relative to M it self. Then we have the following implications for modules:

Monoform modules $\Rightarrow$ Quasi-Dedekind modules $\Rightarrow$ Quasi-monoform modules

The following proposition gives characterizations of relative quasi-invertible submodules.

Proposition (1.1) :[5]

Let N be a submodule of an R-module M. Consider the following.

1. N is a quasi- invertible submodule relative to a submodule P of M with $N \subset P$.
2. Every R-homomorphism $f : P \rightarrow M$ with $f(N)=0$ is trivial,
3. For each $m_1 \in P$, $m_2 \in M$ with $m_2 \neq 0$, there exists $r \in R$ such that $m_1 r \in N$ and $m_2 r \neq 0$.

Then (3) $\Rightarrow$ (2) $\Leftrightarrow$ (1), and (2) $\Rightarrow$(3) if M is injective relative to P.

In section two, we constructed the rational closure of submodule, and then use it to defined rational closed submodule which is a generalization of closed submodule. An R-module M is extending if each submodule is essential in a direct summand, this equivalent to saying that every closed submodule of M is a direct summand. In section three, the rational closure (and hence the rational closed submodule) is the basic ideal to introduced the rationally extending modules which are stronger than that of extending modules. Several properties of rationally extending modules are investigated.

Rational closure of submodules

Let M be an R-module. For any submodule N of M, a closure of N in M is a submodule K of M which is maximal in the family of submodules H of M which contains N as an essential submodule. A submodule K of M is called closed in M if K has no proper essential extensions in M.

Given any submodule N of M, By a complement of N in M we mean a submodule L of M which is maximal in the family of submodules H with the property $N \cap H=0$. A submodule L is called complement in M, if there is a submodule N of M such that L is a complement of N [6]. It is well known that a submodule K of M is closed if and only if K is a complement in M.

Let M be an R-module and let N be a submodule of M. The approach to rational closure of N in M that is given below is not new. The essential ideas of them have appeared regularly in the literature (for instance in [1]). We shall do some variations or modifications on them. The rational extension of submodule N of M refers to any R-homomorphism $\alpha : N \rightarrow M$ such that $\alpha(N)$ is rational in M. In this case, we say that α is rational R-monomorphism. A rational extension is called proper extension of N in case $\alpha(N)$ is a proper submodule of M.

Let M be an R-module and let N be a submodule of M. Set T = $End_R(E(M))$ and S=$Hom_R(E(N)\cap M,E(M))$. Then S is the left ideal of a ring T. In general, E(N) may not be contained in E(M), but E(M) contains a copy of E(N). Thus it may be happen that E(N) has no non-zero elements in common with M, in this case we utilities the isomorphic copy W say of E(N). As E(M) is essential extension of M, hence W ∩ M ≠ 0. Therefore without loss of generality if we assume E(N) ∩ M is non-zero. Now, we formulate the following:

Define RC(N) $=\bigcap_{\alpha}\ker(\alpha)$ where the intersection runs over all elements α in S with N⊆ ker(α). It is clear that N⊆ RC(N).

The following gives some properties of RC(N).

Theorem (2.1): Let M, N, T and S as above. Then:
(a). N is a rational submodule of RC(N), and all submodules of E(N) ∩ M which are rational extensions of N are contained in RC(N),
(b). If N is a rational submodule of B, then the intersection map N → RC(N) extends to R-monomorphism B → RC(N),
(c). RC(N) has no proper rational extension.
Proof:
(a). Let N ⊆ P⊆ RC(N) and f :P → RC(N) be an R-homomorphism with N ⊆ ker(f). Then $i\circ f:P\to E(N)$ where I is the inclusion map of RC(N) into E(N). $i\circ f$ can be extending to g∈S with N ⊆ ker(g). By definition, RC(N) ⊆ ker(g) and hence f(P) = g(P) = 0. Thus N is rational in RC(N). Now, let K be a rational extension of N in E(N) ∩ M and consider α∈S with N⊆ ker(α). Put $K'=K\cap\alpha^{-1}(K)$, then $N\subseteq K'\subseteq K$, hence α restricts to an R-homomorphism $\alpha':K'\to K$ with α' = 0. As N is rational in K, then α' = 0. Thus K ∩ α(K) = $\alpha'(K')$ =0. But K is essential in E(N), and hence K is essential in E(N) ∩ M. Thus α(K) =0 and this implies that K ⊆ RC(N).
(b). The inclusion map i : N→ E(N) can be extended to an R-homomorphism f : B → E(N). Since N is rational in B, then N is essential in B. This implies that f is an R-monomorphism. Not that N ≅ f(N) which is rational in f(B), then by part (a), f(B) ⊆ RC(N) and hence f : B → RC(N).
(c). Let α : RC(N) → C be a rational R-monomorphism. Then α(N) is rational in C and hence α(N) is essential in C. Thus the isomorphism θ : α(N) → N extends to an R-monomorphism β : C → E(N). Since N = βα(N) is rational in β(C), then by part (a), β(C) ⊆ RC(N). Thus αβ: C → C and we note that (αβ-1) α(N) =0. Since α(N) is rational in C, this

implies that $\alpha\beta$-1 = 0 and hence C = $\alpha\beta(C) \subseteq \alpha(RC(N))$. Therefore α is not proper rational extension of RC(N).

Corollary (2.2): Let M be an R-module. For each submodule N of M there exists a submodule P of M such that N is rational in P and P has no proper rational extension.

The following theorem gives a characterization of RC(N).

Theorem (2.3) : Let N be a submodule of an R-module M and S = $Hom_R(E(N) \cap M, E(M))$. Then:

RC(N) = { $x\in E(N) \cap M$ \ for each $y(\neq 0)\in M$, there is $r\in R$ with $xr\in N$ and $yr\neq 0$ }

Proof: Let L = { $x\in E(N) \cap M \setminus \forall y(\neq 0) \in M, \exists\, r\in R$ with $xr\in N$ and $yr\neq 0$ } and let $x\in L$. For any $\alpha : E(N) \cap M \to E(M)$ with $\alpha(N) = 0$, suppose that $y = \alpha(x) \neq 0$, then there exists $r\in R$ such that $xr\in N$ and $0\neq yr=\alpha(xr)$, a contradiction. Thus $\alpha(x)=0$ and $x\in RC(N)$. Conversely, if $x\in RC(N)$ and $x\notin L$, then there exists a non-zero element $y\in E(M)$ such that for each $r\in R$ with $xr\in N$ implies $yr=0$. Define $\psi : N+ xR\to E(M)$ by $\psi(n+xr)=yr$ for each $n\in N$. ψ is well defined non-zero R-homomorphism. Injectivity of E(M) implies that ψ can be extended to $\theta \in S$, a contradiction.

For the proof of the following lemma see [5].

Lemma (2.4): Let M be an R-module and let N be a submodule of E(M). If N is quasi-invertible relative to a submodule P of E(M) with $N\subset P$, then $N\cap M$ is quasi-invertible relative to a submodule $P\cap M$ of M with $N\cap M \subset P\cap M$.

The last theorem asserts that, if N is a submodule of an R-module M then by proposition(1.1), N is quasi-invertible relative to a submodule RC(N) of E(M) with $N\subset RC(N)$. It follows by lemma (2.4) that N is quasi-invertible relative to a submodule RC(N) N is quasi-invertible relative to RC(N) of M with $N\subset RC(N)$. Then we have the following.

Theorem (2.5): Let M be an R-module. For each submodule N of M, there is a submodule P of M such that N is quasi-invertible relative to P and P has no proper rational extension.

Now, we give the following definition.

Definition (2.6): Let N be a submodule of an R-module M. Then:

(1). RC(N) is called the maximal rational extension of N in M. Any maximal rational extension of N in M is isomorphic to RC(N).

(2). N is called rationally closed in M if N=RC(N).

Condition (2) above states that a submodule N of an R-module M is rationally closed if and only if N has no proper rational extension in M.

Note that RC(N) is always rationally closed. It is clear that every closed submodule of an R-module is rationally closed, but the converse may not be true for example see [4]. However for non-singular modules they are equivalent. As every direct summand is closed, hence every direct summand is rationally closed. This motivates to the converse.

Rationally Extending modules

Recall that an R-module M is called extending (or CS-module), if each submodule of M is essential in a direct summand. This is equivalent to saying that every closed (or complement) submodule of M is direct summand.

Definition (3.1): An R-module M is called rationally extending (or RCS-module), if each submodule of M is rational in a direct summand.

It is clear that every rationally extending modules is extending modules, and every monoform modules is trivially rationally extending. Non-singular extending modules are rationally extending.

Proposition (3.2): An R-module M is called rationally extending if and only if each rationally closed submodule of M is direct summand.

Proof: Let N be a rationally closed submodule of M. Then there is a direct summand K of M such that N is rational in K. But N has no proper rational extension in M, then N=K. Conversely, let N be a submodule of M. By theorem (2.1), N is rational in RC(N) and RC(N) is rationally closed, then RC(N) is a direct summand. Thus M is rationally extending.

The following proposition can be easily proved.

Proposition (3.3): An R-module M is monoform if and only if M is indecomposable rationally extending module.

Let R = Z, the ring of integers and M = $Z_{P^{\infty}}$. Then M is trivially extending R-module. It is clear that M is indecomposable. We claim that M is not rationally extending. If not then by proposition (3.3), M is monoform, but this is not true, since if we consider the submodule N = Z_P, then there exist $\frac{1}{P}+Z, \frac{1}{P^2}+Z \in M$ with $\frac{1}{P}+Z \neq 0$. For each $r \in R$, if $(\frac{1}{P^2}+Z)r \in N$ then r is a multiple of P and hence $(\frac{1}{P}+Z)r = 0$. This shows that N is non-zero submodule of M which is not rational in M. Also we can use a similar arguments to show that Z_n is an extending Z-module which is not rationally extending if n is a power of prime numbers for example Z_8 and Z_9, Z_{16} and etc...

Proposition (3.4): Let M be an R-module and let N be a submodule of M. If N is rationally closed in a direct summand of M, Then N is rationally closed in M.

Proof: Let $M=M_1\oplus M_2$ and N is rationally closed submodule of M_1. Assume that N is rational in B for some submodule B of M. Let $\rho: M\rightarrow M_1$ be the projection of M onto M_1. We claim that $\rho(N)$ is rational in

$\rho(B)$. For each $\rho(x)$, $\rho(y)\in \rho(B)$ with $\rho(y)\neq 0$, there exists $r\in R$ such that $\rho(x)r \in N$ and $\rho(y)r \neq 0$. Noting that $\rho(x)r = \rho(\rho(x)r) \in \rho(N)$. Thus $N=\rho(N)$ is rational in $\rho(B) \subseteq M_1$. Since N is rationally closed in M_1, then $\rho(B) = N \subseteq B$ and so $(1-\rho)(B) = B$. Since $(1-\rho)(B)\cap N = 0$ and N is essential in B, then $(1-\rho)(B) = 0$ and hence $B = \rho(B) \subset M_1$ then N = B, since N is rationally closed in M_1.

Corollary (3.5): Direct summands of rationally extending R-module are rationally extending.

Proof: Let N be a direct summand of rationally extending R-module M.If K is a rationally closed submodule of N, then proposition (3.4) implies that K is rational closed in M. Since M is rationally extending, then K is a direct summand of M. Modular law implies that K is a direct summand of N and hence N is rationally extending.

As we have mentioned in the introduction, an R-module M is quasi-monoform if each non-zero proper submodule of M is quasi-invertible relative to a submodule of M. Now we introduce a stronger case as in the following definition.

Definition (3.5): An R-module M is called strongly quasi-monoform if each non-zero proper submodule of M is quasi-invertible relative to some direct summand of M.

Then we can extend the implications mentioned in the introduction to the following one for modules.

Monoform $\Rightarrow$ Quasi-Dedekind $\Rightarrow$ strongly quasi-monoform $\Rightarrow$ quasi-monoform

$\Downarrow$ $\Downarrow$

Rationally extending $\Rightarrow$ strongly quasi-monoform

$\Downarrow$

Extending

Remark: The proof of "every rationally extending module is strongly quasi-monoform" is follows from proposition (1.1).

Recall that an R-module M is multiplication if each submodule N of M has the form N = MB for some ideal A of R. The following theorem is proved in [5].

Theorem (3.6): Let M be a multiplication R-module and N be a submodule of M. If N is quasi-invertible relative to a submodule P of M with $N\subset P$, then N is rational in P.

Theorem (3.7): Let M be a multiplication R-module. Then M is rationally extending module if and only if M is strongly quasi-monoform.

Proof: Suppose that M is rationally extending and N be a non-zero submodule of M, then there exist a direct summand K of M with N is rational in K. Proposition (1.1) implies that N is quasi-invertible relative to K and hence M is strongly quasi-monoform. Conversely, let N be a submodule of strongly quasi-monoform R-module M. Then there is a direct summand K of M such that N is quasi-invertible relative to K. Theorem (3.6) implies that N is rational in K, that is N is rational in a direct summand of M. Hence M is rationally extending.

As every commutative ring R is a multiplication R-module, then we have the following.

Corollary (3.8): A commutative ring R is rationally extending if and only if R is strongly quasi-monoform.

In the following proposition we consider conditions under which extending modules are rationally extending within strongly quasi-monoform modules. First recall the following which appears in [5].

Proposition (3.9): Let M be a multiplication R-module with prime annihilator in R and N, P be submodules of M with $N \subset P$. If N is essential in P, then N is quasi-invertible relative to P.

Theorem (3.10): Let M be a multiplication R-module with prime annihilator in R. Then the following statements are equivalent.

(1). M is extending module

(2). M is strongly quasi-monoform module.

(3). M is rationally extending module.

Proof: (1) $\Rightarrow$(2): follows from proposition (3.9).

(2) $\Rightarrow$(3): follows from theorem (3.7).

(3) $\Rightarrow$(1): It is clear.

Corollary (3.11): Let M be a faithful multiplication module over an integral domain R. Then the following are equivalent.

(1). M is extending module

(2). M is strongly quasi-monoform module.

(3). M is rationally extending module.

In the following we consider another type of conditions under which strongly quasi-monoform module being rationally extending.

Theorem (3.12): Let M be an R-module which is injective relative to all their direct summands.Then M is strongly quasi-monoform module if and only if M is rationally extending.

Proof: Assume that M is strongly quasi-monoform module and N be a submodule of M. Then there is a direct summand P of M such that N is quasi-invertible relative to the submodule P of M with $N \subset P$. Since M is injective relative to P, so by proposition (1.1), for each $m_1 \in P$, $m_2 \in M$ with $m_2 \neq 0$, there exists $r \in R$ such that $m_1 r \in N$ and $m_2 r \neq 0$. This property is a more general of that N is rational in P, and hence M is rationally extending module. The converse is obvious.

Corollary (3.13): Let M be quasi-injective R-module. Then M is strongly quasi-monoform if and only if M is rationally extending module.

Note that Z-module Z_{p^∞} is quasi-injective which is neither rationally extending nor strongly quasi-monoform

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