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STABILITY ANALYSIS OF A COMPETITIVE ECOLOGICAL SYSTEM IN A POLLUTED ENVIRONMENT

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Abstract: The interplay of species in a polluted environment is one of the most critical aspects of the ecosystem. This paper explores the dynamics of the two-species Lokta–Volterra competition model. According to the type I functional response, one species is affected by environmental pollution. Whilst the other degrades the toxin according to the type II functional response. All equilibrium points of the system are located, with their local and global stability being assessed. A numerical simulation examination is carried out to confirm the theoretical results. These results illustrate that competition and pollution can significantly change the coexistence and extinction of each species.

Keywords: polluted environment; competition interaction; local stability; global stability; local bifurcation.

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1. INTRODUCTION

Ecosystems are the result of interactions between the environment and communities. The best method to understand the dynamics and behaviour of ecological interactions between species is to utilise a mathematical model. The earlier ecological interactions description model goes back to

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Lotka and Volterra, now identified as the Lotka – Volterra model [1]-[2].

External effects such as over-predation, over-competition interaction, over-harvesting and pollution lead to the loss of some species [3]–[5].

Today, Toxic pollution is one of the most significant problems confronting the biosphere. Due to this toxicity, the extinction of population species and biodiversity decreases. Thus, it is essential to assess environmental toxicity and evaluate the risk of species in a polluted atmosphere [6]. Organisms are regularly exposed to toxicant environments and absorb toxicants, and pollution endangers the survival of affected populations. [7]. Therefore, we must assess the hazard of the inhabitants exposed to toxicants. So, it is vital to shed light on the impacts of toxicants on populations and find the key-value determining a community's extinction or persistence. Recently, some studies have been made on toxicants emitted from household sources and industries on biological species [8]–[12]. For instance, Liu, Chen, and Zhang looked at a single-species system in a closed toxicant environment with polluted pulse input at a fixed moment. They determined that the inhabitants are extinct when the pulse period is less critical. The persistent condition is met, and the unique positive periodic attractor is globally asymptotically stable [13]. Mukherjee offers a model consisting of two species, one affected by environmental pollution. The toxicant causes an increase in mortality for the first species, while the second species reduce the toxin. He has proven that the system confesses positive global solutions under random fluctuation [7].

In many papers, competition interaction has received scholars' attention[14]–[16]. In particular, a mathematical model has been proposed to describe the interaction among two competing predators-one prey [14]. It has been concluded that Hopf bifurcation could happen when the consumption rate of the second predator is selected as a bifurcation parameter.

This paper proposes the result of a polluted environment on two competitive species in the case of continual emissions from external sources. The two species compete with each other according to Lotka-Volterra type functional responses. Further, it is assumed that the first species uptake pollutants from the environment and negatively affect the growth rate. The second species absorbs the contaminants but is not affected. The rest of this paper is set up as follows: Section 2 investigates the proposed model's assumptions. In section 3, the existence of the possible equilibrium points is found. Then, in section 4, the stability conditions of the steady states have been analysed. In section 5, the global stability of equilibriums is discussed. Further, in section 6, the local bifurcation near the fixed points is established. Finally, some numerical examinations are provided in section 7 to confirm our analytical result.

2. MATHEMATICAL MODEL

Suppose two species compete according to Lokta–Volterra type functional response in a poisoned environment. Type I functional response is used to describe the first species' negative effects due to the environment's pollution. Whilst the other degrades the toxin according to the type II functional response. According to the logistic growth rate form, each species grows independently. Based on assumptions, $s_1(t)$ and $s_2(t)$ are the densities of the two species at the time t. $p(t)$ is the quantity of the contaminant in the atmosphere. Under the above assumptions, the following ODEs are formulated:

$$
\begin{aligned}
\frac{ds_1}{dt} &= r_1 s_1\left(1-\frac{s_1}{k}\right) - \alpha_1 s_1 s_2 - \beta_1 s_1 p = s_1 f_1(s_{1,}s_{2,}p),\\
\frac{ds_2}{dt} &= r_2 s_2\left(1-\frac{s_2}{l}\right) - \alpha_2 s_1 s_2 = s_2 f_2(s_{1,}s_{2,}p), \qquad (1)\\
\frac{dp}{dt} &= r_3 p - dp - \frac{\alpha_3 s_2 p}{\gamma+p} - \beta_2 s_1 p = p f_3(s_{1,}s_{2,}p).
\end{aligned}
$$

All above parameters $\in (0,\infty)$. Further, system (1) has been analysed with the initial values (s_{01}, s_{02}, p_0), where $s_{01} \geq 0, s_{02} \geq 0, p_0 \geq 0$. The flow graph of the system (1) is exposed in the following block diagram.

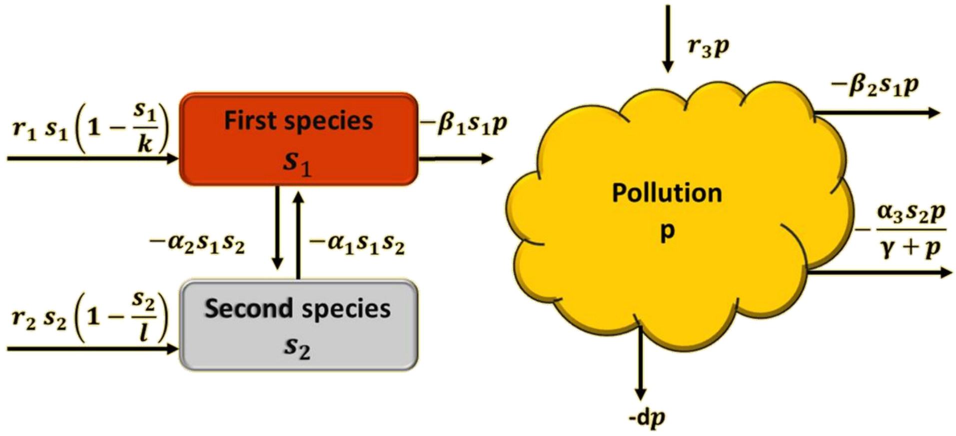

Figure 1 Block diagram for system (1)

We assume that the two species reproduce logistically with the intrinsic growth rates r_1 and r_2 with the carrying capacities k and l, respectively; α_1, α_2 represent computed effect; $\beta_1$is the decay rate of the first species due to pollution; r_3 is the production rate of the toxicant into the surrounding outer sources. d is the reduction rate coefficient of poisonous; α_3 is the uptake rate of toxicants by s_2 with half-saturation constant γ; β_2 is the uptake rate of toxicants by s_1.

The equations on the right-hand side of the system (1) are $C^1(R_+^3)$ on $R_+^3 = \{(s_1, s_2, p), s_1 \geq 0, s_2 \geq 0,\ p \geq 0\}$. Consequently, they are Lipschitzian. Therefore, the system's (1) solution exists and is unique. Further, the model (1) solutions with non-negative initial values remain positive and bounded, as examined in the following section

3. Positivity and Boundedness of the Solutions

Theorem 1. All system's (1) solutions $s_1(t), s_2(t)$ and $p(t)$ of the system (1) with the initial conditions $(s_{01}, s_{02}, p_0) \in R_+^3$ are positively invariant.

Proof. By integrating the interaction function of system (1) for $s_1(t), s_2(t)$ and $p(t)$, we get

$$s_1(t) = s_{01}\, exp\left\{\int_0^t \left[r_1\left(1 - \frac{s_1(s)}{k}\right) - \alpha_1 s_2(s) - \beta_1 p(s)\right] ds\right\},$$

$$s_2(t) = s_{02}\, exp\left\{\int_0^t \left[r_2\left(1 - \frac{s_2(s)}{l}\right) - \alpha_2 s_1(s)\right] ds\right\},$$

$$p(t) = p_0\, exp\left\{\int_0^t \left[r_3 - d - \frac{\alpha_3 s_2(s)}{\gamma + p(s)} - \beta_2 s_1(s)\right] ds\right\}.$$

Then $s_1 \geq 0, s_2 \geq 0$ and $p \geq 0$ for all $t > 0$. Hence the interior of R_+^3 is an invariant set of the system (1).

Theorem 2. All solutions $s_1(t), s_2(t)$ and $p(t)$ of the system (1) with the initial values (s_{01}, s_{02}, p_0) are uniformly bounded.

Proof: - Let $\left(s_1(t), s_2(t), p(t)\right)$ be an arbitrary system (1) solution with a non-negative initial condition. Then for $N(t) = s_1(t) + s_2(t) + p(t)$, we obtain

$$\frac{dN}{dt} = \frac{ds_1}{dt} + \frac{ds_2}{dt} + \frac{dp}{dt}$$

i.e.,

$$\frac{dN}{dt} = r_1 s_1 - \frac{r_1 s_1^2}{k} - \alpha_1 s_1 s_2 - \beta_1 s_1 p + r_2 s_2 - \frac{r_2 s_2^2}{l} - \alpha_2 s_1 s_2 + r_3 p - dp - \frac{\alpha_3 s_2 p}{\gamma + p} - \beta_2 s_1 p$$

Hence, according to the assumptions of the theorem, the following is obtained:

$\frac{dN}{dt} \leq r_1 s_1 + r_2 s_2 + r_3 p - dp,$

$\frac{dN}{dt} + \sigma_1 N \leq 2r_1 s_1 + 2r_2 s_2 + 2r_3 p.$

Where $\sigma_1 = min.\{r_1 + r_2 + (r_3 + d)\}$, then

$\frac{dN}{dt} + \sigma_1 N \leq 2r_1 s_1 + 2r_2 s_2 + 2r_3 p = 2\sigma_2$

Applying Gromwell's Inequality, the following is obtained:

$$0 \leq N(s_1(t), s_2(t), p(t)) \leq \frac{2\sigma_2}{\sigma_1}(1 - e^{-\sigma_1 t}) + H(0)e^{-\sigma_1 t}$$

hence,

$$0 \leq \limsup_{t \to \infty} N(t) \leq \frac{2\sigma_2}{\sigma_1}.$$

Thus, all system's (1) solutions that are initiated in R_+^3 are attracted to the region $\vartheta = \left\{(s_1, s_2, p) \in R_+^3 : N = s_1 + s_2 + p \leq \frac{2\sigma_2}{\sigma_1}\right\}$. Thus, these solutions are uniformly bounded.

4. Existence of Equilibria

System (1) has eight non-negative steady states, namely

(1) The disappearing equilibrium point $I_1 = (0,0,0)$.

(2) The first species equilibrium point $I_2 = (k, 0,0)$.

(3) The second species equilibrium point $I_3 = (0, l, 0)$.

(4) The species' free equilibrium point $I_4 = (0,0,\tilde{p})$, where is any positive real number.

(5) The first free species equilibrium point $I_5 = (0, l, \check{p})$, where $\check{p} = \frac{\alpha_3 l}{(r_3 - d)} - \gamma > 0$ if and only if

$$\alpha_3 l > \gamma(r_3 - d). \tag{2}$$

(6) The second free species equilibrium point $I_6 = (\bar{s}_1, 0, \bar{p})$, where $\bar{s}_1 = \frac{r_3 - d}{\beta_2}$ and $\bar{p} = \frac{r_1}{k\beta_1\beta_2}\left(k\beta_2 - (r_3 - d)\right)$. It should be noted that for $\bar{s}_1$ and $\bar{p}$ to be positive, the following must

be the case

$$0 < (r_3 - d) < k\beta_2. \tag{3}$$

(7) The pollution free equilibrium point $I_7 = (\hat{s}_1, \hat{s}_2, 0)$, where $\hat{s}_1 = \frac{r_2 k(\alpha_1 l - r_1)}{\alpha_1 \alpha_2 lk - r_1 r_2}$ and $\hat{s}_2 = \frac{r_1}{\alpha_1}\left[1 - \frac{r_2(\alpha_1 l - r_1)}{\alpha_1 \alpha_2 lk - r_1 r_2}\right]$. Clearly, $\hat{s}_1 > 0$, if one of the following conditions hold:

$$r_1 < min.\left\{\alpha_1 l, \frac{\alpha_1 \alpha_2 lk}{r_2}\right\}, \tag{4}$$

$$r_1 > max.\left\{\alpha_1 l, \frac{\alpha_1 \alpha_2 lk}{r_2}\right\}. \tag{5}$$

Further, $\hat{s}_2 > 0$ if the following holds:

$$r_2(\alpha_1 l - r_1) < (\alpha_1 \alpha_2 lk - r_1 r_2). \tag{6}$$

(8) The positive equilibrium point $I_8 = ({s_1}^*, {s_2}^*, p^*)$, where ${s_1}^* = \frac{r_2}{\alpha_2}(1 - \frac{{s_2}^*}{l})$, $p^* = \frac{r_1}{\beta_1} - \frac{r_1 r_2}{\beta_1 k \alpha_2} + \frac{{s_2}^*}{\beta_1}\left(\frac{r_1 r_2}{lk\alpha_2} - \alpha_1\right)$ and ${s_2}^*$ is the root of the following equation:

$$As_2^2 + Bs_2 + C = 0. \tag{7}$$

Here $A = \frac{\beta_2 r_2(r_1 r_2 - \alpha_1 \alpha_2 lk)}{\alpha_2^2 l^3 \beta_1}$, $B = \left(r_3 - d - \frac{\beta_2 r_2}{\alpha_2}\right)\left(\frac{r_1 r_2 - \alpha_1 \alpha_2 lk}{\alpha_2 \beta_1 kl}\right) + \frac{\gamma \beta_2 r_2}{\alpha_2 l} + \frac{r_1 r_2 \beta_2(k\alpha_2 - r_2)}{\alpha_2^2 \beta_1 kl}$, $C = \left(r_3 - d - \frac{\beta_2 r_2}{\alpha_2}\right)\left(\gamma + \frac{r_1(k\alpha_2 - r_2)}{\alpha_2 \beta_1 k}\right)$. Using Descartes's rule of sign Eq. (7) has a unique positive root if the sign of B and C are the same and opposite to the sign of A, or if the sign of A and B are the same and opposite to the sign of C. That means one of the following conditions must be the case:

1. $A > 0, B > 0$ and $C < 0$,
2. $A < 0, B < 0$ and $C > 0$,
3. $B > 0, C > 0$ and $A < 0$,
4. $B < 0, C < 0$ and $A > 0$.

Further, for ${s_1}^*$and p^* to be positive, the following must be the case

$$\frac{lr_1(r_2 - k\alpha_2)}{r_1 r_2 - \alpha_1 \alpha_2 lk} < {s_2}^* < l. \tag{8}$$

5. LOCAL STABILITY

This section explores the local stability behaviour of system (1) 's equilibrium points.

The Jacobin matrix of system (1) at any point, say (s_1, s_2, p), can be written as:

$$J = \begin{bmatrix} s_1\frac{\partial f_1}{\partial s_1} + f_1 & s_1\frac{\partial f_1}{\partial s_2} & s_1\frac{\partial f_1}{\partial p} \\ s_2\frac{\partial f_2}{\partial s_1} & s_2\frac{\partial f_2}{\partial s_2} + f_2 & s_2\frac{\partial f_2}{\partial p} \\ p\frac{\partial f_3}{\partial s_1} & p\frac{\partial f_3}{\partial s_2} & p\frac{\partial f_3}{\partial p} + f_3 \end{bmatrix} = (a_{ij})_{3\times 3},$$

where, $a_{11} = r_1 - \frac{2r_1s_1}{k} - \alpha_1 s_2 - \beta_1 p$; $a_{12} = -\alpha_1 s_1$; $a_{13} = -\beta_1 s_1$; $a_{21} = -\alpha_2 s_2$; $a_{22} = r_2 - \frac{2r_2s_2}{l} - \alpha_2 s_1$; $a_{23} = 0$; $a_{31} = -\beta_2 p$; $a_{32} = -\frac{\alpha_3 p}{\gamma+p}$; $a_{33} = r_3 - d - \frac{\gamma\alpha_3 s_2}{(\gamma+p)^2} - \beta_2 s_1$.

1. The Jacobian matrix at I_1= (0,0,0) is given as:

$$J(I_1) = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 - d \end{bmatrix} \tag{9}$$

Then, $J(I_1)$ has the eigenvalues $\lambda_{11} = r_1 > 0$, $\lambda_{12} = r_2 > 0$, and $\lambda_{13} = r_3 - d$, which means I_1 is unstable if $r_3 > d$. Further, I_1 is a saddle point when $r_3 < d$.

2. The Jacobian matrix at $I_2 = (k, 0,0)$ is given as:

$$J(I_2) = \begin{bmatrix} -r_1 & -\alpha_1 k & -\beta_1 k \\ 0 & r_2 - \alpha_2 k & 0 \\ 0 & 0 & r_3 - d - \beta_2 k \end{bmatrix}. \tag{10}$$

Then, $J(I_2)$ has the eigenvalues $\lambda_{21} = -r_1 < 0$, $\lambda_{22} = r_2 - \alpha_2 k$ and $\lambda_{23} = r_3 - d - \beta_2 k$. I_2 is a locally asymptotical stable point, if and only if the following condition is satisfied:

$$k > max.\left\{\frac{r_2}{\alpha_2}, \frac{r_3 - d}{\beta_2}\right\}, \tag{11}$$

3. The Jacobian matrix at $I_3 = (0, l, 0)$ is given as:

$$J(I_3) = \begin{bmatrix} r_1 - \alpha_1 l & 0 & 0 \\ -\alpha_2 l & -r_2 & 0 \\ 0 & 0 & r_3 - d - \frac{\alpha_3 l}{\gamma} \end{bmatrix}. \tag{12}$$

Then, $J(I_3)$ has the eigenvalues $\lambda_{31} = r_1 - \alpha_1 l, \lambda_{32} = -r_2 < 0$ and $\lambda_{33} = r_3 - d - \frac{\alpha_3 l}{\gamma}$. That means I_3 is a locally asymptotical stable if and only if the following is satisfied

$$l > max.\left\{\frac{r_1}{\alpha_1}, \frac{(r_3-d)\gamma}{\alpha_3}\right\}. \tag{13}$$

4. The Jacobian matrix at $I_4 = (0,0,\tilde{p})$ is given as:

$$J(I_4) = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ -\beta_2\tilde{p} & \frac{-\alpha_3\tilde{p}}{\gamma+\tilde{p}} & r_3 - d \end{bmatrix}. \tag{14}$$

Then, $J(I_4)$ has the eigenvalues $\lambda_{41} = r_1 > 0$, $\lambda_{42} = r_2 > 0$, and $\lambda_{43} = r_3 - d$, which means I_4 is unstable if $r_3 > d$. Further, I_4 is a saddle point when $r_3 < d$.

5. The Jacobian matrix at $I_5 = (0, l, \check{p})$ is given as:

$$J(I_5) = \begin{bmatrix} r_1 - \alpha_1 l - \beta_1\check{p} & 0 & 0 \\ -\alpha_2 l & -r_2 & 0 \\ -\beta_2\check{p} & \frac{\alpha_3\check{p}}{\gamma+\check{p}} & r_3 - d - \frac{\alpha_3\gamma l}{(\gamma+\check{p})^2} \end{bmatrix}. \tag{15}$$

Then, $J(I_5)$ has the eigenvalues $\lambda_{51} = r_1 - \alpha_1 l - \beta_1\check{p}$, $\lambda_{52} = -r_2 < 0$ and $\lambda_{53} = r_3 - d - \frac{\alpha_3\gamma l}{(\gamma+\check{p})^2}$. That means I_5 is locally asymptotically stable if

$$l > max.\left\{\frac{r_1-\beta_1\check{p}}{\alpha_1}, \frac{(r_3-d)(\gamma+\check{p})^2}{\alpha_3\gamma}\right\}. \tag{16}$$

6. The Jacobian matrix at $I_6 = (\bar{s}_1, 0, \bar{p})$ is given as:

$$J(I_6) = \begin{bmatrix} \frac{-r_1(r_3-d)}{k\beta_2} & -\alpha_1\bar{s}_1 & -\beta_1\bar{s}_1 \\ 0 & r_2 - \alpha_2\bar{s}_1 & 0 \\ -\beta_2\bar{p} & \frac{\alpha_3\bar{p}}{\gamma+\bar{p}} & 0 \end{bmatrix}. \tag{17}$$

Then, the characteristic equation of $J(I_6)$ is given by:

$$(r_2-\alpha_2\bar{s}_1-\lambda)\left(\lambda^2+\frac{r_1(r_3-d)}{k\beta_2}\lambda-\beta_1\beta_2\bar{s}_1\bar{p}\right). \quad (18)$$

The eigenvalues of Eq. (18) can be written as follows $\lambda_{62}=r_2-\alpha_2\bar{s}_1,\ \lambda_{61}+\lambda_{63}=\frac{-r_1(r_3-d)}{k\beta_2}<0$ and $\lambda_{61}.\lambda_{63}=-\beta_1\beta_2\bar{s}_1\bar{p}<0$. That means I_6 is a saddle point.

7. The Jacobian matrix at $I_7=(\hat{s}_1,\hat{s}_2,0)$ is given as:

$$J(I_7)=\begin{bmatrix}\frac{-r_1\hat{s}_1}{k} & -\alpha_1\hat{s}_1 & -\beta_1\hat{s}_1\\ -\alpha_2\hat{s}_2 & \frac{-r_2\hat{s}_2}{l} & 0\\ 0 & 0 & r_3-d-\frac{\alpha_3\hat{s}_2}{\gamma}-\beta_2\hat{s}_1\end{bmatrix}. \quad (19)$$

Then, the characteristic equation of $J(I_7)$ is given by:

$$\left(r_3-d-\frac{\alpha_3\hat{s}_2}{\gamma}-\beta_2\hat{s}_1-\lambda\right)\left[\lambda^2+\frac{(lr_1\hat{s}_1+kr_2\hat{s}_2)}{kl}\lambda+\hat{s}_1\hat{s}_2\left(\frac{r_1r_2}{kl}-\alpha_1\alpha_2\right)\right]. \quad (20)$$

The eigenvalues of Eq. (20) can be written as follows $\lambda_{73}=r_3-d-\frac{\alpha_3\hat{s}_2}{\gamma}-\beta_2\hat{s}_1,\ \lambda_{71}+\lambda_{72}=\frac{-(lr_1\hat{s}_1+kr_2\hat{s}_2)}{kl}<0$ and $\lambda_{71}.\lambda_{72}=\hat{s}_1\hat{s}_2\left(\frac{r_1r_2}{kl}-\alpha_1\alpha_2\right)$.

That means I_7 is locally asymptotically stable if

$$r_3<d+\frac{\alpha_3\hat{s}_2}{\gamma}+\beta_2\hat{s}_1, \quad (21)$$

$$r_1r_2>\alpha_1\alpha_2kl. \quad (22)$$

8. The Jacobian matrix at $I_8=({s_1}^*,{s_2}^*,p^*)$ is given as:

$$J(I_8)=\begin{bmatrix}\frac{-r_1s_1^*}{k} & -\alpha_1{s_1}^* & -\beta_1{s_1}^*\\ -\alpha_2{s_2}^* & \frac{-r_2{s_2}^*}{l} & 0\\ -\beta_2p^* & \frac{-\alpha_3p^*}{\gamma+p^*} & r_3-d-\frac{(\gamma+p^*)(\alpha_3{s_2}^*)-(\alpha_3p^*{s_2}^*)}{(\gamma+p^*)^2}-\beta_2{s_1}^*\end{bmatrix}=\left(b_{ij}\right)_{3\times3}, \quad (23)$$

where, $b_{11}=\frac{-r_1s_1^*}{k},\ b_{12}=-\alpha_1s_1^*,\ b_{13}=-\beta_1s_1^*,\ b_{21}=-\alpha_2s_1^*,\ b_{22}=\frac{-r_2s_2^*}{l},\ b_{23}=0,\ b_{31}=$

$-\beta_2 p^*$, $b_{32} = \frac{-\alpha_3 p^*}{\gamma+p^*}$, and $b_{33} = r_3 - d - \frac{\gamma\alpha_3 s_2^*}{(\gamma+p^*)^2} - \beta_2 s_1^*$.

So, the characteristic equation of $J(I_8)$ can be written as:

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0, \tag{24}$$

where,

$A_1 = -(m_1 + b_{33})$,

$A_2 = b_{22}b_{33} - m_2 - m_3$,

$A_3 = m_2 b_{33} + b_{13}b_{31}b_{22}$,

$\Delta = A_1A_2 - A_3 = -m_1m_2 + (b_{11} + b_{33})m_3 + b_{22}b_{33}A_1 - b_{11}b_{22}b_{33}$.

Here, $m_1 = b_{11} + b_{22} < 0$, $m_2 = b_{12}b_{21} - b_{11}b_{22}$ and $m_3 = b_{13}b_{31} - b_{11}b_{33}$.

Now, according to the Routh-Hurwitz criteria [18], all the eigenvalues of $J(I_8)$ have roots with negative real parts, on condition that $A_1 > 0, A_3 > 0$ and $\Delta > 0$. Then, is a locally asymptotical stable point if the following conditions are satisfied

$$\frac{-b_{13}b_{31}b_{22}}{m_2} < b_{33} < min.\left\{\frac{-m_1m_2+(b_{11}+b_{33})m_3+b_{22}b_{33}A_1}{b_{11}b_{22}}, -(b_{11} + b_{22})\right\}. \tag{25}$$

6. GLOBAL DYNAMICAL BEHAVIOUR

This section discusses the conditions of the global stability property of the system's (1) equilibria using the Lyapunov method.

Theorem 3. Assume that $I_2 = (k, 0,0)$ is exist., then the basin of attraction of I_2 is the sub-region of R^3_+ which can be defined as**:** $\emptyset_1 = \{(s_1, s_2, p): s_1 \geq \frac{(d-r_3)\alpha_1+kr_2\beta_1}{\beta_1}, s_2 \geq 0, p \geq 0\}$.

Proof: Define $w_2 = c_1\left(s_1 - k - k\,ln\frac{s_1}{k}\right) + c_2 s_2 + c_3 p$, where c_1, c_2 and c_3 are positive constants to be determined. $w_2(s_1, s_2, \text{p})$ is a positive definite about I_2. Thus,

$$\frac{dw_2}{dt} = c_1(s_1 - k)\left[-\frac{r_1 s_1}{k} - \alpha_1 s_2 - \beta_1 p + r_1\right] + c_2 s_2\left[r_2\left(1 - \frac{s_2}{l}\right) - \alpha_2 s_1\right] + c_3 p\left[r_3 - d - \frac{\alpha_3 s_2}{\gamma + p} - \beta_2 s_1\right]$$

i.e.,

$$\frac{dw_2}{dt}=\frac{-c_1r_1(s_1-k)^2}{k}-c_1\alpha_1s_1s_2+c_1k\alpha_1s_2-c_1\beta_1s_1p+c_1\beta_1kp+c_2r_2s_2-\frac{c_2r_2s_2^2}{l}$$
$$-c_2\alpha_2s_1s_2+c_3p(r_3-d)-\frac{c_3\alpha_3ps_2}{\gamma+p}-c_3\beta_2s_1p.$$

By choosing the positive constant as: $c_1=\frac{d-r_3}{k\beta_1}$, $c_2=c_3=1$, the following is obtained,

$$\frac{dw_2}{dt}=-\frac{r_2s_2^2}{l}-\frac{r_1(d-r_3)(s_1-k)^2}{\beta_1k^2}-\frac{(d-r_3)\alpha_1s_1s_2}{k\beta_1}+\frac{(d-r_3)\alpha_1s_2}{\beta_1}-\frac{(d-r_3)s_1p}{k}+r_2s_2$$
$$-\alpha_2s_1s_2-\frac{\alpha_3ps_2}{\gamma+p}-\beta_2s_1p.$$

Then, $\frac{dw_2}{dt}<0$ if the reduction rate coefficient of the toxicant is greater than its production rate. Hence, w_2 is a Lyapunov function. Therefore, any solution stating from $\emptyset_1$ approach asymptotically to I_2.

Theorem 4. Assume that $I_3=(0,l,0)$ is exist, then the basin of attraction of I_3 is the sub-region of R_+^3 which can be defined as: $\emptyset_2=\left\{(s_1,s_2,p): s_1\geq\left\{\frac{(r_1+\alpha_2l)k}{r_1}\right\}, s_2>0, p\geq 0\right\}$.

Proof: Define $w_3=s_1+\left(s_2-l-l\ln\frac{s_2}{l}\right)+p$, where $w_3(s_1,s_2,\mathrm{p})$ is a positive definite about I_3. Thus,

$$\frac{dw_3}{dt}==s_1\left[r_1\left(1-\frac{s_1}{k}\right)-\alpha_1s_2-\beta_1p\right]+(s_2-l)\left[\frac{-r_2s_2}{l}-\alpha_2s_1+r_2\right]$$
$$+p\left[r_3-d-\frac{\alpha_3s_2}{\gamma+p}-\beta_2s_1\right]$$

i.e.,

$$\frac{dw_3}{dt}=s_1r_1-\frac{r_1s_1^2}{k}-\alpha_1s_1s_2-\beta_1s_1p-\frac{r_2(s_2-l)^2}{l}-\alpha_2s_1s_2+l\alpha_2s_1+(r_3-d)p$$
$$-\frac{\alpha_3ps_2}{\gamma+p}-\beta_2ps_1$$

Then, $\frac{dw_3}{dt}<0$ if the production rate of the toxicant is less than its reduction coefficient rate. Hence, w_3 is a Lyapunov function. Therefore, any solution stating from $\emptyset_2$ approach asymptotically to I_3.

Theorem 5. Assume that $I_5=(0,l,\check{p})$ exist, then the basin of attraction of I_5 is the sub-region of R_+^3 which can be defined as**:** $\emptyset_3=\{s_1,s_2,p): s_1>0, s_2\geq\frac{r_1+\alpha_2 l+\check{p}\beta_2}{\alpha_2}, p=\check{p}\}$.

Proof: Define $w_4=s_1+\left(s_2-l-l\ln\frac{s_2}{l}\right)+\left(p-\check{p}-\check{p}\ln\frac{p}{\check{p}}\right)$, where $w_4(s_1,s_2,\mathrm{p})$ is a positive definite about I_5. Thus,

$$\frac{dw_4}{dt}=s_1\left[r_1\left(1-\frac{s_1}{k}\right)-\alpha_1 s_2-\beta_1 p\right]+(s_2-l)\left[\frac{-r_2 s_2}{l}-\alpha_2 s_1+r_2\right]$$
$$+(p-\check{p})\left[-\frac{\alpha_3 s_2}{\gamma+p}-\beta_2 s_1+\frac{\alpha_3 l}{\gamma+\check{p}}\right].$$

Therefore,

$$\frac{dw_4}{dt}=s_1(r_1-\alpha_2 s_2+l\alpha_2+\beta_2\check{p})-\frac{\alpha_3 s_2(p-\check{p})}{\gamma+p}+\frac{\alpha_3 l(p-\check{p})}{\gamma+\check{p}}-\beta_2 s_1 p-\frac{r_1 s_1^2}{k}-\frac{r_2(s_2-l)^2}{l}.$$

Then, $\frac{dw_4}{dt}<0$ in $\emptyset_3$. Hence, w_4 is a Lyapunov function. Therefore, any solution stating from $\emptyset_3$ approach asymptotically to I_5.

Theorem 6. Suppose that the following conditions are satisfied

$$kl(\alpha_1+\alpha_2)^2\leq 4r_1 r_2, \tag{26}$$

$$d>r_3. \tag{27}$$

Then $I_7=(\hat{s}_1,\hat{s}_2,0)$ is globally asymptotically stable in R_+^3.

Proof: Define $w_6=c_1\left(s_1-\hat{s}_1-\hat{s}_1\ln\frac{s_1}{\hat{s}_1}\right)+c_2\left(s_2-\hat{s}_2-\hat{s}_2\ln\frac{s_2}{\hat{s}_2}\right)+c_3 p$, where c_1, c_2 and c_3 are positive constants to be determined. $w_6(s_1,s_2,\mathrm{p})$ is a positive definite about I_7. Thus,

$$\frac{dw_6}{dt}=c_1(s_1-\hat{s}_1)\left[-\frac{r_1 s_1}{k}-\alpha_1 s_2-\beta_1 p+\frac{r_1\hat{s}_1}{k}+\alpha_1\hat{s}_2\right]$$
$$+c_2(s_2-\hat{s}_2)\left[-\frac{r_2 s_2}{l}-\alpha_2 s_1+\frac{r_2\hat{s}_2}{l}+\alpha_2\hat{s}_1\right]+c_3 p\left[r_3-d-\frac{\alpha_3 s_2}{\gamma+p}-\beta_2 s_1\right].$$

Therefore,

$$\frac{dw_6}{dt}=-\frac{c_1 r_1}{k}(s_1-\hat{s}_1)^2-(c_1\alpha_1+c_2\alpha_2)(s_1-\hat{s}_1)(s_2-\hat{s}_2)-\frac{c_2 r_2}{L}(s_2-\hat{s}_2)^2-c_1\beta_1 p(s_1$$
$$-\hat{s}_1)+c_3 p(r_3-d)-\frac{c_3\alpha_3 p s_2}{\gamma+p}-c_3\beta_2 p s_1.$$

By choosing the positive constants as: $c_1 = c_2 = 1, c_3 = \frac{\beta_1 \hat{s}_1}{d - r_3}$, the following is obtained,

$$\frac{dw_6}{dt} \leq -\left[\sqrt{\frac{r_1}{k}}(s_1 - \hat{s}_1) + \sqrt{\frac{r_2}{l}}(s_2 - \hat{s}_2)\right]^2 - \beta_1 p s_1 - \frac{\alpha_3 \beta_1 \hat{s}_1 p s_2}{(d - r_3)(\gamma + p)} - \frac{\beta_1 \beta_2 \hat{s}_1 p s_1}{d - r_3}.$$

Then, $\frac{dw_6}{dt} < 0$ under conditions (26)-(27). Hence, w_6 is a Lyapunov function. Therefore, I_7 is globally asymptotically stable in R^3_+.

Theorem 7. Assume that

$$r_1 \geq max.\left\{\frac{kl(\alpha_1+\alpha_2)^2}{r_2}, \frac{(\beta_1+\beta_2)^2}{\alpha_3 s_2^*}, \frac{k\alpha_3(\gamma+p^*)}{s_2^*}\right\}, \tag{28}$$

then $I_8 = (s_1^*, s_2^*, p^*)$ is globally asymptotically stable in R^3_+.

Proof: - Define $w_7 = \left(s_1 - s_1^* - s_1^* \ln\frac{s_1}{s_1^*}\right) + \left(s_2 - s_2^* - s_2^* \ln\frac{s_2}{s_2^*}\right) + \left(p - p^* - p^* \ln\frac{p}{p^*}\right)$ where $w_7(s_1, s_2, \mathrm{p})$ is a positive definite about I_8. Thus,

$$\frac{dw_7}{dt} = (s_1 - s_1^*)\left[-\frac{r_1 s_1}{k} - \alpha_1 s_2 - \beta_1 p + \frac{r_1 s_1^*}{k} + \alpha_1 s_2^* + \beta_1 p^*\right] + (s_2 - s_2^*)\left[-\frac{r_2 s_2}{l} - \alpha_2 s_1 + \frac{r_2 s_2^*}{l} + \alpha_2 s_1^*\right] + (p - p^*)\left[-\frac{\alpha_3 s_2}{\gamma+p} - \beta_2 s_1 + \frac{\alpha_3 l}{\gamma+p^*} + \beta_2 s_1^*\right]$$

Then,

$$\frac{dw_7}{dt} = -\frac{r_1(s_1 - s_1^*)^2}{k} - (\alpha_1 + \alpha_2)(s_1 - s_1^*)(s_2 - s_2^*) - \frac{r_2(s_2 - s_2^*)^2}{l} - (\beta_1 + \beta_2)(s_1 - s_1^*)(p - p^* + \alpha_3 s_2^*(p - p^*)^2 - \beta_2(p - p^*)(s_1 - s_1^*) + \alpha_3(\gamma + p^*)(s_2 - s_2^*)(p - p^*)$$

i.e.,

$$\frac{dw_7}{dt} = \left[-\frac{r_1(s_1 - s_1^*)^2}{2k} - (\alpha_1 + \alpha_2)(s_1 - s_1^*)(s_2 - s_2^*) - \frac{r_2(s_2 - s_2^*)^2}{2l}\right] - \left[\frac{r_1(s_1 - s_1^*)^2}{2k} + (\beta_1 + \beta_2)(s_1 - s_1^*)(p - p^*) + \frac{\alpha_3 s_2^*(p - p^*)^2}{2}\right] - \left[\frac{r_2(s_2 - s_2^*)^2}{2l} + \alpha_3(\gamma + p^*)(s_2 - s_2^*)(p - p^*) + \frac{\alpha_3 s_2^*(p - p^*)^2}{2}\right].$$

Therefore, the following is obtained

$$\frac{dw_7}{dt} \leq -\left[\sqrt{\frac{r_1}{2k}}(s_1 - s_1^*)^2 + \sqrt{\frac{r_2}{2l}}(s_2 - s_2^*)\right]^2 - \left[\sqrt{\frac{r_1}{2k}}(s_1 - s_1^*) + \sqrt{\frac{\alpha_3 s_2^*}{2}}(p - p^*) - \right]^2$$

$$-\left[\sqrt{\frac{r_2}{2l}}(s_2 - s_2^*) + \sqrt{\frac{\alpha_3 s_2^*}{2}}(p - p^*)\right]^2.$$

Then, $\frac{dw_7}{dt} < 0$ under condition (28). Hence, w_7 is a Lyapunov function. Therefore, I_8 is globally asymptotically stable in R_+^3.

7. The Persistence

Persistence signifies a global property in a dynamic system. Biologically, it means the survival of all system populations in future times. While mathematically implies that strictly positive solutions do not have an omega limit set on the boundary of the non-negative cone. In contrast, the system populations threaten extinction if one loses persistence.

The average Lyapunov function method is used to explore the system's (1) persistence. But first, the boundary planes' global behaviour needs to be studied. Clearly, system (1) has the following two sub-systems

1. The sub-system in $s_1 s_2$-plane

$$\begin{aligned} h_1 &= s_1\left[r_1\left(1 - \frac{s_1}{k}\right) - \alpha_1 s_2\right] \\ h_2 &= s_2\left[r_2\left(1 - \frac{s_2}{l}\right) - \alpha_2 s_1\right] \end{aligned} \tag{29}$$

2. In $s_2 p$-plane

$$\begin{aligned} h_3 &= s_2\left[r_2(1 - \frac{s_2}{l})\right] \\ h_4 &= p\left[r_3 - d - \frac{\alpha_3 s_2}{\gamma + p}\right] \end{aligned} \tag{30}$$

Both sub-systems have strictly positive equilibria in the positive quadrant of the $s_1 s_2$-plane and $s_2 p$ - plane, which is illustrated by a projection of the boundary planar steady states (s_1, s_2) and (s_2, p) of (29) and (30).

Now, define the function $\mathrm{H}(s_1, s_2) = \frac{1}{s_1 s_2}$, which is $C^1(R_+^2)$ in $R^2_{+(s_1,s_2)} = \{(s_1, s_2), s_1 > 0, s_2 > 0,\}$, thus $\Delta(s_1, s_2) = \frac{\partial}{\partial s_1}(\mathrm{H}h_1) + \frac{\partial}{\partial s_2}(\mathrm{H}h_2) = \frac{-r_1}{k s_2} - \frac{s_2}{l s_1} < 0$. This means $\Delta(s_1, s_2)$ does not change sign and is not identically zero.

Therefore, the (29) has no periodic dynamics in $R^2_{+(s_1,s_2)}$. Then the strictly positive equilibrium point is globally asymptotically stable whenever it exists. Using the same strategy, it is concluded that (30) has no periodic dynamics in $R^2_{+(s_2,p)}$.

Theorem 8. Assume that (29) and (30) have no periodic dynamics, then system (1) is uniformly persistent if

$$l < min.\left\{\frac{r_1 - \beta_1 \check{p}}{\alpha_1}, \frac{(r_3 - d)\gamma}{\alpha_3}\right\}, \tag{31}$$

$$k < min.\left\{\frac{r_2}{\alpha_2}, \frac{(r_3 - d)}{\beta_2}\right\}, \tag{32}$$

$$\hat{s}_1 < min.\left\{\frac{(r_1 - \alpha_1 \hat{s}_2)k}{r_1}, \frac{r_2 l - r_2 \hat{s}_2}{l \alpha_2}, \frac{r_3 \gamma - d\gamma - \alpha_3 \hat{s}_2}{\gamma \beta_2}\right\}. \tag{33}$$

Proof. Define $\emptyset(s_1, s_2, p) = s_1^a s_2^b p^c$, where a, b and c are positive constants. Clearly $\emptyset(s_1, s_2, \mathrm{p}) > 0$ for all $(s_1, s_2, \mathrm{p}) \in R_+^3$ and $\emptyset(s_1, s_2, \mathrm{p}) \to 0$ when one of the variables s_1, s_2 or p approaches zero.

Consequently,

$$\delta(s_1, s_2, \mathrm{p}) = \frac{\emptyset'(s_1, s_2, \mathrm{p})}{\emptyset(s_1, s_2, \mathrm{p})} = s_1\left[r_1\left(1 - \frac{s_1}{k}\right) - \alpha_1 s_2\right] + s_2\left[r_2\left(1 - \frac{s_2}{l}\right) - \alpha_2 s_1\right] + p\left[r_3 - d - \frac{\alpha_3 s_2}{\gamma + p}\right].$$

Now, the only possible omega limit sets of the system (1) on the boundary of $s_1 s_2 p-$plane is the equilibrium points I_2, I_3, I_5 and I_7. Thus,

$\delta(I_2) = \mathrm{a}[r_1 - \alpha_1 l] + \mathrm{c}\left[r_3 - d - \frac{\alpha_3 l}{\gamma}\right] > 0$ under condition (31).

$\delta(I_3) = b[r_2 - \alpha_2 k] + \mathrm{c}[r_3 - d - \beta_2 k] > 0$ under condition (32).

$\delta(I_5) = a[r_1 - \alpha_1 l - \beta_1 \check{p}] + \mathrm{c}\left[r_3 - d - \frac{\alpha_3 l}{\gamma + \check{p}}\right] > 0$ under condition (31).

$\delta(I_7) = \mathrm{a}\left[r_1 - \frac{r_1 \hat{s}_1}{k} - \alpha_1 \hat{s}_2\right] + \mathrm{b}\left[r_2 - \frac{r_2 \hat{s}_2}{l} - \alpha_2 \hat{s}_1\right] + \mathrm{c}\left[r_3 - d - \frac{\alpha_3 \hat{s}_2}{\gamma} - \beta_2 \hat{s}_1\right] > 0$ under condition (33). Hence system (1) is uniformly persistent.

8. LOCAL BIFURCATION

This section uses Sotomayor's theorem to study the local bifurcation conditions near the steady states.

Now, the Jacobian matrix of system (1) at each of the equilibrium points is given by:

$$J=\begin{bmatrix} r_1-\frac{2r_1s_1}{k}-\alpha_1s_2-\beta_1p & -\alpha_1s_1 & -\beta_1s_1 \\ -\alpha_2s_2 & r_2-\frac{2r_2s_2}{l}-\alpha_2s_1 & 0 \\ -\beta_2p & -\frac{\alpha_3p}{\gamma+p} & r_3-d-\frac{\gamma\alpha_3s_2}{(\gamma+p)^2}-\beta_2s_1 \end{bmatrix}$$

For nonzero vector $X=(x_1,x_2,x_3)^T$:

$$D^2F(x,x)=\begin{bmatrix} \frac{-2r_1x_1^2}{k}-2\alpha_1x_1x_2-2\beta_1x_1x_3 \\ -2\alpha_2x_1x_2-\frac{2r_2x_2^2}{l} \\ -2\beta_2x_1x_3-\frac{2\gamma\alpha_3x_2x_3}{(\gamma+p)^2}-\frac{2\gamma\alpha_3s_2x_3^2}{(\gamma+p)^3} \end{bmatrix}, \tag{34}$$

The following theorems determine the saddle-node bifurcation of the system (1) at the equilibrium point I_2 .

Theorem 9. For the $r_2^*=\alpha_2k$, system (1), at the equilibrium point I_2 has a saddle-node bifurcation if

$$lr_1\alpha_2\neq k\alpha_1r_2^*. \tag{35}$$

Proof: - According to $J(I_2)$, given by (10), system (1), at the equilibrium point I_2, has a zero eigenvalue, say λ_{22}, at $r_2^*=\alpha_2k$, and the Jacobian matrix $J^*(I_2)=J(I_2,r_2^*)$, becomes:

$$J^*(I_2)=\begin{bmatrix} -r_1 & -\alpha_1k & -\beta_1k \\ 0 & 0 & 0 \\ 0 & 0 & r_3-d-\beta_2k \end{bmatrix}$$

Now, let $X^{[2]}=\left(x_1^{[2]},x_2^{[2]},x_3^{[2]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{22}=0$. Thus $(J^*(I_2)-\lambda_{22}I)X^{[2]}=0$, which gives:

$x_2^{[2]}=\frac{-\alpha_1kx_1^{[2]}}{r_1}, x_3^{[2]}=0$ and $x_1^{[2]}$ represents any nonzero real number. That means $X^{[2]}=$

$\left(x_1^{[2]}, x_2^{[2]}, 0\right)^T$.

Let $Y^{[2]} = \left(y_1^{[2]}, y_2^{[2]}, y_3^{[2]}\right)^T$ be an eigenvector associated with the eigenvalue λ_{22} of the matrix J_2^{*T}. Then $(J_2^{*T} - \lambda_{22} I)Y^{[2]} = 0$. By solving this equation for $Y^{[2]}$, $Y^{[2]} = \left(0, y_2^{[2]}, 0\right)^T$ is obtained, where $y_2^{[2]}$ is any nonzero real number.

Now, to check that the conditions of Sotomayor's theorem for saddle-node bifurcation are satisfied, the following is measured:

$$\frac{\partial F}{\partial r_2} = F_{r_2}(S, r_2) = \left(\frac{\partial f_1}{\partial r_2}, \frac{\partial f_2}{\partial r_2}, \frac{\partial f_3}{\partial r_2}\right)^T = \left(0, 1 - \frac{s_2}{l}, 0\right)^T.$$

So, $F_{r_2} = (I_2, r_2^*) = (0,1,0)^T$ and hence $\left(Y^{[2]}\right)^T F_{r_2}(I_2, r_2^*) = y_2^{[2]} \neq 0$.

Therefore, the first condition of the saddle-node bifurcation is met whilst transcortical, and pitchfork bifurcation cannot occur.

Subsequently,

$$D^2 F_{r_2}(I_2, r_2{}^*)\left(x^{[2]}, x^{[2]}\right) = \left(\frac{-2r_1\left[x_1^{[2]}\right]^2}{k} - 2\alpha_1 x_1^{[2]} x_2^{[2]}, -2\alpha_2 x_1^{[2]} x_2^{[2]} - \frac{2r_2^*\left[x_2^{[2]}\right]^2}{l}, 0\right)^T,$$

hence, it is obtained that:

$\left(Y^{[2]}\right)^T \left[D^2 F_{r_2}(I_2, r_2^*)\left(X^{[2]}, X^{[2]}\right)\right] = -2y_2^{[2]} x_1^{[2]} x_2^{[2]} \left[\alpha_2 - \frac{k\alpha_1 r_2^*}{l r_1}\right] \neq 0$ under condition (35). This means the second condition of saddle-node bifurcation is satisfied. Thus, according to Sotomayor's theorem, system (1) has saddle-node bifurcation at I_2 with the parameter $r_2{}^* = \alpha_2 k$.

Theorem 10. For $\alpha_1^* = \frac{r_1}{l}$, system (1), at the equilibrium point I_3 has a saddle-node bifurcation if

$$r_1 r_2 \neq k l \alpha_1^* \alpha_2. \tag{36}$$

Proof: - According to $J(I_3)$, given by (12), system (1), at I_3 , has a zero eigenvalue, say λ_{31}, when $\alpha_1^* = \frac{r_1}{l}$, and the Jacobian matrix $J^*(I_3) = J(I_3, \alpha_1^*)$, becomes:

$$J^*(I_3) = \begin{bmatrix} 0 & 0 & 0 \\ -\alpha_2 l & -r_2 & 0 \\ 0 & 0 & r_3 - d - \dfrac{\alpha_3 l}{\gamma} \end{bmatrix}$$

Now, let $X^{[3]} = \left(x_1^{[3]}, x_2^{[3]}, x_3^{[3]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{31} = 0$. Thus $(J^*(I_3) - \lambda_{31} I)X^{[3]} = 0$, which implies: $x_2^{[3]} = \frac{-\alpha_2 l x_1^{[3]}}{r_2}, x_3^{[3]} = 0$ and $x_1^{[3]}$ represents any nonzero real number. That means $X^{[3]} = \left(x_1^{[3]}, x_2^{[3]}, 0\right)^T$.

Let $Y^{[3]} = \left(y_1^{[3]}, y_2^{[3]}, y_3^{[3]}\right)^T$ be an eigenvector associated with the eigenvalue λ_{31} of the matrix $J^*(I_3)$. Then $(J_3^{*T} - \lambda_{31} I)Y^{[3]} = 0$. By solving this equation for $Y^{[3]}$, $Y^{[3]} = \left(y_1^{[3]}, 0, 0\right)^T$ is obtained, where $y_1^{[3]}$ is any nonzero real number.

Now, to check that the conditions of Sotomayor's theorem for saddle-node bifurcation are satisfied, the following is measured:

$$\frac{\partial F}{\partial \alpha_1} = F_{\alpha_1}(S, \alpha_1) = \left(\frac{\partial f_1}{\partial \alpha_1}, \frac{\partial f_2}{\partial \alpha_1}, \frac{\partial f_3}{\partial \alpha_1}\right)^T = (-s_2, 0, 0)^T.$$

So, $F_{\alpha_1} = (I_3, \alpha_1^*) = (-l, 0, 0)^T$ and hence $\left(Y^{[3]}\right)^T F_{\alpha_1}(I_3, \alpha_1^*) = -l y_1^{[3]} \neq 0$.

Therefore, the first condition of the saddle-node bifurcation is met whilst transcortical, and pitchfork bifurcation cannot occur.

Subsequently,

$$D^2 F_{\alpha_1}(I_3, \alpha_1^*)\left(x^{[3]}, x^{[3]}\right) = \left(\frac{-2r_1\left[x_1^{[3]}\right]^2}{k} - 2\alpha_1^* x_1^{[3]} x_2^{[3]}, -2\alpha_2 x_1^{[3]} x_2^{[3]} - \frac{2r_2\left[x_2^{[3]}\right]^2}{l}, 0\right)^T.$$

Hence, $\left(Y^{[3]}\right)^T \left[D^2 F_{\alpha_1}(I_3, \alpha_1^*)\left(X^{[3]}, X^{[3]}\right)\right] = \left(y_1^{[3]}, 0, 0\right)\left(\frac{-2r_1\left[x_1^{[3]}\right]^2}{k} - 2\alpha_1^* x_1^{[3]} x_2^{[3]}, 0, 0\right)^T = -2y_1^{[3]}\left[x_1^{[3]}\right]^2\left[\frac{r_1}{k} - \frac{l\alpha_1^*\alpha_2}{r_2}\right] \neq 0$ under condition (36). This means the second condition of saddle-node bifurcation is satisfied. Thus, according to Sotomayor's theorem, system (1) has saddle-node bifurcation at I_3 with the parameter $\alpha_1{}^* = \frac{r_1}{l}$.

Theorem 11. For $r_1^* = \alpha_1 k + \beta_1 \check{p}$, system (1), at the equilibrium point I_5 has a saddle-node bifurcation if

$$r_1^* x_1^{[5]} + k\alpha_1 x_2^{[5]} + k\beta_1 x_3^{[5]} \neq 0, \tag{37}$$

where $x_i^{[5]}, i = 1,2,3$ are given in the proof.

Proof: - According to the Jacobian matrix $(J(I_5)$, given by (15), system (1), at the equilibrium point I_5, has a zero eigenvalue, say λ_{51} at $r_1^* = \alpha_1 l + \beta_1 \check{p}$, and the Jacobian matrix $J^*(I_5) = J(I_5, r_1^*)$, becomes:

$$J^*(I_5) = \begin{bmatrix} 0 & 0 & 0 \\ -\alpha_2 l & -r_2 & 0 \\ -\beta_2 \check{p} & \dfrac{\alpha_3 \check{p}}{\gamma + \check{p}} & r_3 - d - \dfrac{\alpha_3 \gamma l}{(\gamma + \check{p})^2} \end{bmatrix}$$

Now, let $X^{[5]} = \left(x_1^{[5]}, x_2^{[5]}, x_3^{[5]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{51} = 0$. Thus $(J^*(I_5) - \lambda_{51} I)X^{[5]} = 0$, which gives:
$x_2^{[5]} = \frac{-\alpha_2 l x_1^{[5]}}{r_2}, x_3^{[5]} = \frac{(\gamma+\check{p})[r_2\beta_2\check{p}(\gamma+\check{p}) - \alpha_2\alpha_3 l\check{p}]x_1^{[5]}}{r_2((r_3-d)(\gamma+\check{p}) - \alpha_3\gamma l)}$ and $x_1^{[5]}$ represents any nonzero real number and $(r_3 - d)(\gamma + \check{p}) - \alpha_3 \gamma l \neq 0$.

Let $Y^{[5]} = \left(y_1^{[5]}, y_2^{[5]}, y_3^{[5]}\right)^T$ be an eigenvector associated with the eigenvalue λ_{51} of the matrix J_3^{*T}. Then $(J_5^{*T} - \lambda_{51} I)Y^{[5]} = 0$. By solving this equation for $Y^{[5]}$, $Y^{[5]} = \left(y_1^{[5]}, 0,0\right)^T$ is obtained, where $y_1^{[5]}$ is any nonzero real number.

Now, to check that the conditions of Sotomayor's theorem for saddle-node bifurcation are satisfied, the following is considered:

$$\frac{\partial F}{\partial r_1} = F_{r_1}(S, r_1) = \left(\frac{\partial f_1}{\partial r_1}, \frac{\partial f_2}{\partial r_1}, \frac{\partial f_3}{\partial r_1}\right)^T = \left(1 - \frac{s_1}{k}, 0,0\right)^T.$$

So, $F_{r_1} = (I_5, r_1^*) = (1,0,0)^T$ and hence $\left(Y^{[5]}\right)^T F_{r_1}(I_5, r_1^*) = y_1^{[5]} \neq 0$.

Therefore, the first condition of the saddle-node bifurcation is met whilst transcortical, and pitchfork bifurcation cannot occur.

Subsequently,

$$D^2F_{r_1}(I_5, r_1{}^*)\left(x^{[5]}, x^{[5]}\right)$$

$$= \left(\frac{-2r_1^*\left[x_1^{[5]}\right]^2}{k} - 2\alpha_1 x_1^{[5]}x_2^{[5]} - 2\beta_1 x_1^{[5]}x_3^{[5]}, -2\alpha_2 x_1^{[5]}x_2^{[5]} - \frac{2r_2\left[x_2^{[5]}\right]^2}{l}, -2\beta_2 x_1^{[5]}x_3^{[5]} - \frac{2\gamma\alpha_3 x_2 x_3}{(\gamma+\check{p})^2} - \frac{2\gamma\alpha_3 s_2 x_3^2}{(\gamma+\check{p})^3}\right)^T,$$

hence, it is obtained that:

$$\left(Y^{[5]}\right)^T\left[D^2F_{r_1}(I_5, r_1^*)\left(X^{[5]}, X^{[5]}\right)\right]$$

$$= \left(y_1^{[5]}, 0,0\right)\left(\frac{-2r_1^*\left[x_1^{[5]}\right]^2}{k} - 2\alpha_1 x_1^{[5]}x_2^{[5]} - 2\beta_1 x_1^{[5]}x_3^{[5]}, -2\alpha_2 x_1^{[5]}x_2^{[5]} - \frac{2r_2\left[x_2^{[5]}\right]^2}{l}, -2\beta_2 x_1^{[5]}x_3^{[5]} - \frac{2\gamma\alpha_3 x_2 x_3}{(\gamma+\check{p})^2} - \frac{2\gamma\alpha_3 s_2 x_3^2}{(\gamma+\check{p})^3}\right)^T.$$

i.e.,

$\left(Y^{[5]}\right)^T\left[D^2F_{r_1}(I_5, r_1^*)\left(X^{[5]}, X^{[5]}\right)\right] = -2y_1^{[5]}x_1^{[5]}\left[\frac{r_1^* x_1^{[5]}}{k} + \alpha_1 x_2^{[5]} + \beta_1 x_3^{[5]}\right] \neq 0$ under condition (37). This means the second condition of saddle-node bifurcation is satisfied. Thus, according to Sotomayor's theorem, system (1) has saddle-node bifurcation at I_5 with the parameter r_1^*.

Theorem 12. For $r_3^* = d + \frac{\alpha_3\hat{s}_2}{\gamma} - \beta_2\hat{s}_1$, system (1), at the equilibrium point I_7 has a saddle-node bifurcation if

$$\beta_2 x_1^{[7]} + \frac{\alpha_3 x_2^{[7]}}{\gamma} + \frac{\alpha_3\hat{s}_2 x_3^{[7]}}{\gamma^2} \neq 0, \tag{38}$$

where $x_i^{[7]}, i = 1,2,3$ is given in the proof.

Proof: According to the Jacobian matrix $J(I_7)$, given by (19), system (1), at the equilibrium point I_7, has a zero eigenvalue, say λ_{73} at $r_3^* = d + \frac{\alpha_3\hat{s}_2}{\gamma} + \beta_2\hat{s}_1$, and the Jacobian matrix $J^*(I_7) = J(I_7, r_3^*)$, becomes:

$$J^*(I_7) = \begin{bmatrix} \frac{-r_1\hat{s}_1}{k} & -\alpha_1\hat{s}_1 & -\beta_1\hat{s}_1 \\ -\alpha_2\hat{s}_2 & \frac{-r_2\hat{s}_2}{l} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, let $X^{[7]} = \left(x_1^{[7]}, x_2^{[7]}, x_3^{[7]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{73} = 0$. Thus $(J^*(I_7) - \lambda_{73}I)X^{[7]} = 0$, which gives:

$x_1^{[7]} = \frac{-r_2 x_2^{[7]}}{l\alpha_2} = x_3^{[7]} = \frac{r_1r_2 - kl\alpha_1\alpha_2}{kl\alpha_2\beta_1}$ and $x_2^{[7]}$ represents any nonzero real number and $r_1r_2 - kl\alpha_1\alpha_2 \neq 0$.

Let $Y^{[7]} = \left(y_1^{[7]}, y_2^{[7]}, y_3^{[7]}\right)^T$ be an eigenvector associated with the eigenvalue λ_{73} of the matrix J_7^{*T}. Then $(J_7^{*T} - \lambda_{73}I)Y^{[7]} = 0$. By solving this equation for $Y^{[7]}$, $Y^{[7]} = \left(0,0, y_3^{[7]}\right)^T$ is obtained, where $y_3^{[7]}$ is any nonzero real number.

Now, to check that the conditions of Sotomayor's theorem for saddle-node bifurcation are satisfied, the following is considered:

$$\frac{\partial F}{\partial r_3} = F_{r_3}(S, r_3) = \left(\frac{\partial f_1}{\partial r_3}, \frac{\partial f_2}{\partial r_3}, \frac{\partial f_3}{\partial r_3}\right)^T = (0,0,1)^T.$$

So, $F_{r_3} = (I_7, r_3^*) = (0,0,1)^T$and hence $\left(Y^{[7]}\right)^T F_{r_3}(I_7, r_3^*) = y_3^{[7]} \neq 0$

Therefore, the first condition of the saddle-node bifurcation is met whilst transcortical, and pitchfork bifurcation cannot occur.

Subsequently,

$$D^2F_{r_3}(I_7, {r_3}^*)\left(x^{[7]}, x^{[7]}\right)$$

$$= \left(\frac{-2r_1\left[x_1^{[7]}\right]^2}{k} - 2\alpha_1 x_1^{[7]} x_2^{[7]} - 2\beta_1 x_1^{[7]} x_3^{[7]}, -2\alpha_2 x_1^{[7]} x_2^{[7]} - \frac{2r_2\left[x_2^{[7]}\right]^2}{l}, -2\beta_2 x_1^{[7]} x_3^{[7]} - \frac{2\alpha_3 x_2 x_3}{\gamma} - \frac{2\alpha_3 s_2 x_3^2}{\gamma^2}\right)^T$$

hence, it is obtained that:

$$\left(Y^{[7]}\right)^T\left[D^2F_{r_3}(I_7,r_3^*)\left(X^{[7]},X^{[7]}\right)\right]$$
$$=\left(0,0,y_3^{[7]}\right)\left(\frac{-2r_1\left[x_1^{[7]}\right]^2}{k}-2\alpha_1x_1^{[7]}x_2^{[7]}-2\beta_1x_1^{[7]}x_3^{[7]},-2\alpha_2x_1^{[7]}x_2^{[7]}\right.$$
$$\left.-\frac{2r_2\left[x_2^{[7]}\right]^2}{l},-2\beta_2x_1^{[7]}x_3^{[7]}-\frac{2\alpha_3x_2^{[7]}x_3^{[7]}}{\gamma}-\frac{2\alpha_3s_2\left[x_3^{[7]}\right]^2}{\gamma^2}\right)^T.$$

i.e.,

$$\left(Y^{[7]}\right)^T\left[D^2F_{r_3}(I_7,r_3^*)\left(X^{[7]},X^{[7]}\right)\right]=-2x_3^{[7]}y_3^{[7]}\left[\beta_2x_1^{[7]}+\frac{\alpha_3x_2^{[7]}}{\gamma}+\frac{\alpha_3\hat{s}_2x_3^{[7]}}{\gamma^2}\right]\neq 0.$$

This means the second condition of saddle-node bifurcation is satisfied under condition (38). Thus, according to Sotomayor's theorem, system (1) has saddle-node bifurcation at I_7 with the parameter r_3^*.

Theorem 13. If the parameter r_2 passes through $r_2^*=\frac{l(m_1m_2+(a_{11}a_{33})m_3)}{s_2a_{33}A_1}$, where $r_2^*>0$. Then system (1), at the equilibrium point I_8 has

1) a saddle-node bifurcation provided that

$$l\neq s_2^*, \tag{39}$$

$$\left(Y^{[8]}\right)^T\left[D^2F_{r_2}(I_8,r_2^*)\left(X^{[8]},X^{[8]}\right)\right]\neq 0. \tag{40}$$

2) a transcritical bifurcation if condition (39) is violated while condition (40) is satisfied.
3) a pitchfork bifurcation if conditions (39)-(40) are violated with the following state is satisfied

$$\left(Y^{[8]}\right)^T\left[D^3F_{r_2}(I_8,r_2^*)\left(X^{[8]},X^{[8]}\right)\right]\neq 0. \tag{41}$$

where the formula of $Y^{[8]}$ and $X^{[8]}$ are given in following the proof.

Proof: According to the Jacobian matrix $J(I_8)$, given by (23), system (1), at the equilibrium point I_8, has a zero eigenvalue, say λ_{83} at $r_2^* = \frac{l(m_1m_2+(a_{11}a_{33})m_3)}{s_2a_{33}A_1}$, and the Jacobian matrix $J^*(I_8) = J(I_8, r_2^*)$, becomes:

$$J^*(I_8) = \begin{bmatrix} \frac{-r_1s^*{}_1}{k} & -\alpha_1 s_1^* & -\beta_1 s_1^* \\ -\alpha_2 s_2^* & \frac{-r_2s_2^*}{l} & 0 \\ -\beta_2 p^* & \frac{-\alpha_3 p^*}{\gamma+p^*} & r_3 - d - \frac{\gamma\alpha_3 s_2^*}{(\gamma+p^*)^2} - \beta_2 s_1^* \end{bmatrix}.$$

Now, let $X^{[8]} = \left(x_1^{[8]}, x_2^{[8]}, x_3^{[8]}\right)^T$ be the eigenvector corresponding to the eigenvalue say $\lambda_{83} = 0$. Thus $(J^*(I_8) - \lambda_{83}I)X^{[8]} = 0$, which gives:

$x_1^{[8]} = \frac{-r_2^*x_2^{[8]}}{l\alpha_2} = x_3^{[8]} = \frac{(r_1r_2^*-kl\alpha_1\alpha_2)x_2^{[8]}}{kl\alpha_2\beta_1}$ and $x_2^{[8]}$ represents any nonzero real number. That means

$X^{[8]} = \left(x_1^{[8]}, x_2^{[8]}, x_3^{[8]}\right)^T$.

Let $Y^{[8]} = \left(y_1^{[8]}, y_2^{[8]}, y_3^{[8]}\right)^T$ be an eigenvector associated with the eigenvalue λ_{83} of the matrix J_8^{*T}. Then $(J_8^{*T} - \lambda_{83}I)Y^{[8]} = 0$. By solving this equation for $Y^{[8]}$, $Y^{[8]} = \left(y_1^{[8]}, y_2^{[8]}, y_3^{[8]}\right)^T$ is obtained, where $y_2^{[8]} = \frac{-y_1^{[8]}s_1^*l(\alpha_1q_1(\gamma+p^*)+\alpha_3\beta_1p^*)}{r_2s_2^*q_1(\gamma+p^*)}, y_3^{[8]} = \frac{\beta_1s_1^*y_1^{[8]}}{q_1}$ where $y_1^{[8]}$ is any nonzero real number and $q_1 = r_3 - d - \frac{\gamma\alpha_3s_2^*}{(\gamma+p^*)^2} - \beta_2s_1^* \neq 0$.

Now, to confirm whether the conditions of Sotomayor's theorem for saddle-node bifurcation are satisfied, the following is considered:

Now, consider:

$$\frac{\partial F}{\partial r_2} = DF_{r_2}(S, r_2) = \left(\frac{\partial f_1}{\partial r_3}, \frac{\partial f_2}{\partial r_3}, \frac{\partial f_3}{\partial r_3}\right)^T = \left(0, 1 - \frac{s_2}{l}, 0\right)^T.$$

So, $F_{r_2} = (I_8, r_2^*) = \left(0, 1 - \frac{s_2^*}{l}, 0\right)^T$ and hence $\left(Y^{[8]}\right)^T F_{r_2}(I_8, r_2^*) = y_2^{[8]}\left[1 - \frac{s_2^*}{l}\right] \neq 0$ under condition (39). Therefore, transcortical bifurcation cannot occur whilst the first condition of the saddle-node bifurcation is met. Now,

$$D^2F_{r_2}(I_8,r_2{}^*)\left(x^{[8]},x^{[8]}\right)$$
$$=\left(\frac{-2r_1\left[x_1^{[8]}\right]^2}{k}-2\alpha_1x_1^{[8]}x_2^{[8]}-2\beta_1x_1^{[8]}x_3^{[8]},\qquad -2\alpha_2x_1^{[8]}x_2^{[8]}\frac{2r_2^*\left[x_2^{[8]}\right]^2}{l},\right.$$
$$\left.-2\beta_2x_1^{[8]}x_3^{[8]}-\frac{2\gamma\alpha_3x_2^{[8]}x_3^{[8]}}{(\gamma+p^*)^2}-\frac{2\gamma\alpha_3s_2^*\left[x_3^{[8]}\right]^2}{(\gamma+p^*)^3}\right)^T.$$

Hence, it is obtained that:

$$\left(Y^{[8]}\right)^T\left[D^2F_{r_2}(I_8,r_2^*)\left(X^{[8]},X^{[8]}\right)\right]$$
$$=\left(y_1^{[8]},y_2^{[8]},y_3^{[8]}\right)\left(\frac{-2r_1\left[x_1^{[8]}\right]^2}{k}-2\alpha_1x_1^{[8]}x_2^{[8]}-2\beta_1x_1^{[8]}x_3^{[8]},\right.$$
$$\left.-2\alpha_2x_1^{[8]}x_2^{[8]}-\frac{2r_2^*\left[x_2^{[8]}\right]^2}{l},-2\beta_2x_1^{[8]}x_3^{[8]}-\frac{2\gamma\alpha_3x_2^{[8]}x_3^{[8]}}{(\gamma+p^*)^2}-\frac{2\gamma\alpha_3s_2^*\left[x_3^{[8]}\right]^2}{(\gamma+p^*)^3}\right)^T$$

i.e.,

$\left(Y^{[8]}\right)^T\left[D^2F_{r_2}(I_8,r_2^*)\left(X^{[8]},X^{[8]}\right)\right]=\frac{-2r_1y_1^{[8]}\left[x_1^{[8]}\right]^2}{k}-2\alpha_1y_1^{[8]}x_1^{[8]}x_2^{[8]}-2\beta_1y_1^{[8]}x_1^{[8]}x_3^{[8]}-2\alpha_2y_2^{[8]}x_1^{[8]}x_2^{[8]}-\frac{2r_2^*y_2^{[8]}\left[x_2^{[8]}\right]^2}{l}-2\beta_2y_3^{[8]}x_1^{[8]}x_3^{[8]}-\frac{2\gamma\alpha_3x_2^{[8]}x_3^{[8]}y_3^{[8]}}{(\gamma+p^*)^2}-\frac{2\gamma\alpha_3s_2^*y_3^{[8]}\left[x_3^{[8]}\right]^2}{(\gamma+p^*)^3}\neq 0$ under condition (40). This means the second condition of saddle-node bifurcation is satisfied.

Moreover, if condition (39) is not satisfied, then the following is obtained:

$$\left(Y^{[8]}\right)^T F_{r_2}(I_8,r_2^*)=y_2^{[8]}\left[1-\frac{s_2^*}{l}\right]=0.$$

So, according to Sotomayor's theorem, saddle-node bifurcation cannot occur while the first condition of transcritical bifurcation is satisfied.

Now,

$$DF_{r_2}(S,r_2)=\begin{bmatrix}0&0&0\\0&\frac{-1}{l}&0\\0&0&0\end{bmatrix}$$

where, $DF_{r_2}(S, r_2)$ represents the derivative of $F_{r_2}(S, r_2)$ with respect to $S = (s_1, s_2, p)^T$. Furthermore, it is observed that:

$$DF_{r_2}(I_8, r_2^*)X^{[8]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{-1}{l} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1^{[8]} \\ x_2^{[8]} \\ x_3^{[8]} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{-x_2^{[8]}}{l} \\ 0 \end{bmatrix}$$

$$\left(Y^{[8]}\right)^T DF_{r_2}(I_8, r_2^*)X^{[8]} = \left(y_1^{[8]}, y_2^{[8]}, y_3^{[8]}\right)\left(0, \frac{-x_2^{[8]}}{l}, 0\right)^T = \frac{-y_2^{[8]}x_2^{[8]}}{l} \neq 0$$

Hence, according to condition (40):

$$\left(Y^{[8]}\right)^T \left[D^2F_{r_2}(I_8, r_2^*)\left(X^{[8]}, X^{[8]}\right)\right] \neq 0$$

This means the required conditions to have transcritical bifurcation are satisfied.

Finally, if conditions (39)-(40) are not satisfied, then the following is obtained:

$\left(Y^{[8]}\right)^T F_{r_2}(I_8, r_2^*) = y_2^{[8]}\left[1 - \frac{s_2^*}{l}\right] = 0$ and $\left(Y^{[8]}\right)^T\left[D^2F_{r_2}(I_8, r_2^*)\left(X^{[8]}, X^{[8]}\right)\right] = 0$. So, according to Sotomayor's theorem, the first and second conditions of pitchfork bifurcation are satisfied.

Now,

$$D^3F(X, X) = \begin{bmatrix} 0 \\ 0 \\ \frac{2\gamma\alpha_3 x_2^{[8]}\left[x_3^{[8]}\right]^2}{(\gamma + p)^3} + \frac{6\gamma\alpha_3 s_2\left[x_3^{[8]}\right]^3}{(\gamma + p)^4} \end{bmatrix}.$$

Hence, according to condition (41):

$$\left(Y^{[8]}\right)^T\left[D^3F_{r_2}(I_8, r_2^*)\left(X^{[8]}, X^{[8]}\right)\right] = \left(y_1^{[8]}, y_2^{[8]}, y_3^{[8]}\right)\left[0, 0, \frac{2\gamma\alpha_3 x_2^{[8]}\left[x_3^{[8]}\right]^2}{(\gamma + p^*)^3} + \frac{6\gamma\alpha_3 s_2\left[x_3^{[8]}\right]^3}{(\gamma + p^*)^4}\right]^T$$

$$= \frac{2\gamma\alpha_3 x_2^{[8]}y_3^{[8]}\left[x_3^{[8]}\right]^2}{(\gamma + p^*)^2}\left[\frac{1}{(\gamma + p^*)} - \frac{3s_2^*(r_1r_2^* - kl\alpha_1\alpha_2)}{kl\alpha_2\beta_1(\gamma + p^*)^2}\right] \neq 0$$

This means system (1) has pitchfork bifurcation at I_8 with the parameter r_2^*.

Theorem 14. Assume that the following conditions are satisfied

$$max.\left\{\frac{m_2+m_3}{b_{22}}, \frac{-b_{13}b_{31}b_{22}}{m_2}\right\} < b_{33} < -(b_{11}+b_{22}), \tag{42}$$

$$A_3'(r_1^*) \neq \left(A_1'(r_1^*)A_2(r_1^*) + A_1(r_1^*)A_2'(r_1^*)\right), \tag{43}$$

$$r_1^* > 0, \tag{44}$$

where Ai's are the coefficients of the characteristic equation given in Eq. (24), and the formula of r_1^* is shown in the following proof. Then, system (1) has a Hopf bifurcation at $r_1 = r_1^*$ for I_8.

Proof: - Consider the characteristic equation which gives in (24) at I_8. Now, to verify the conditions for a Hopf bifurcation to occur, we need to find a parameter such that $\Delta = A_1A_2 - A_3 = 0$ is satisfied. It is observed that $\Delta = 0$ gives:

$$r_1^* = \frac{-k(m_1m_2+(b_{11}+b_{11})m_3+b_{22}b_{33}A_1)}{b_{22}b_{33}s_1^*}.$$

Clearly, $r_1^* > 0$ provided that the condition (44) holds. Now, at $r_1 = r_1^*$ the characteristic equation given by Eq. (24) can be written as

$$(\lambda + A_1)(\lambda^2 + A_2) = 0, \tag{45}$$

which has three roots

$$\lambda_1 = -A_1, \lambda_{2,3} = \pm i\sqrt{A_2}.$$

Clearly, at $r_1 = r_1^*$ there are two purely imaginary eigenvalues λ_2 and λ_3 and one eigenvalue λ_1 which have negative real parts. Now for all values of r_1 in the neighbourhood of r_1^*, the roots in general, have the following forms:

$$\lambda_1 = -A_1, \lambda_{2,3} = \delta_1(r_1) \pm i\delta_2(r_1).$$

Clearly, $Re(\lambda_{2,3})|_{r_1=r_1^*} = \delta_1(r_1^*) = 0$ means the first condition for Hopf bifurcation is satisfied at $r_1 = r_1^*$. Now to verify the transversality condition, we substitute $\delta_1(r_1) \pm i\delta_2(r_1)$ into Eq. (45), and then calculate its derivative with respect to the bifurcation parameter r_1^*, $\Theta(r_1^*)\psi(r_1^*) + \Gamma(r_1^*)\phi(r_1^*) \neq 0$, where the form of $\Theta(r_1^*), \psi(r_1^*), \Gamma(r_1^*)$ and $\phi(r_1^*)$ are

$\psi(r_1) = 3\delta_1^2(r_1) + 2A_1(r_1)\delta_1(r_1) + A_2(r_1) - 3\delta_2^2(r_1),$

$\phi(r_1) = 6\delta_1(r_1)\delta_2(r_1) + 2A_1(r_1)\delta_2(r_1),$

$\Theta(r_1) = \delta_1^2(r_1)A_1{}'(r_1) + A_2{}'(r_1)\delta_1(r_1) + A_3{}'(r_1) - A_1{}'(r_1)\delta_2^2(r_1),$

$\Gamma(r_1) = 2\delta_1(r_1)\delta_2(r_1)A_1'(r_g) + A_2'(r_1)\delta_2(r_1).$

Note that for $r_1 = r_1^*$, we have $\delta_1 = 0$ and $\delta_2 = \sqrt{A_2}$, substitution into Eq. (45) gives the following simplifications:

$$\begin{aligned}
\psi(r_1^*) &= -2A_2(r_1^*),\\
\phi(r_1^*) &= 2A_1(r_1^*)\sqrt{A_2(r_1^*)},\\
\Theta(r_1^*) &= A_3'(r_1^*) - A_1'(r_1^*)A_2(r_1^*),\\
\Gamma(r_1^*) &= A_2'(r_1^*)\sqrt{A_2(r_1^*)},
\end{aligned}$$

where,

$$\begin{aligned}
A_1'(r_1^*) &= \frac{s_1^*}{k},\\
A_2'(r_1^*) &= \frac{-s_1^*}{k}(a_{22} + a_{33}),\\
A_3'(r_1^*) &= \frac{s_1^* a_{33}^2}{k}.
\end{aligned}$$

Hence, condition (43) gives

$$\Theta(r_1^*)\psi(r_1^*) + \Gamma(r_1^*)\phi(r_1^*) = -2A_2(r_1^*)\left[A_3'(r_1^*) - \left(A_1'(r_1^*)A_2(r_1^*) + A_1(r_1^*)A_2'(r_1^*)\right)\right] \neq 0.$$

This means that Hopf bifurcation has occurred.

9. Numerical Analysis

Numerical simulations of the system (1) are obtained to demonstrate the analytical results of our study. The dynamics of the model (1) are carried out through the help of MATLAB. Then, the time series and phases diagram of the solutions of system (1) are drawn for the following set of parameters:

$$r_1 = 65, r_2 = 20, r_3 = 10, k = 4, l = 5,, \alpha_1 = 0.025, \alpha_2 = 0.6,\ \beta_1 = 0.03, \beta_2 = 0.005, \alpha_3 = 0.05,\ d = 1, \gamma = 0.05. \tag{46}$$

For different sets of initial values (3,5,1), (3,3,3) and (2.6,1,1), the system's (1) solution approaches asymptotically to the globally stable point $I_8 = (3.9,4.4,19.6)$ (see Figure 1).

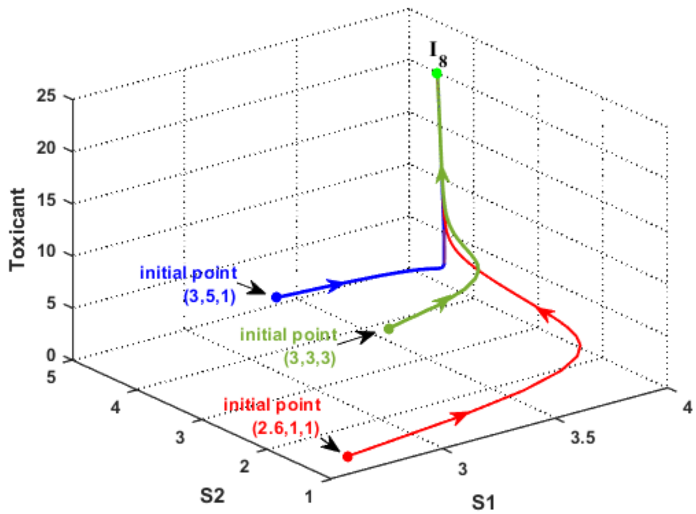

Figure 2 Phase diagram of system (1) with the data given by (46) with different initial values.

Model (1) is now numerically resolved for the data in (46) to investigate the impact of altering one parameter at a time on system's (1) behaviour. For this purpose, Figure 2 depicts the dynamics of the two species with the set of data given by (46), with different values of β_1. It illustrates the solution of system (1) settles down to I_8 for different values of β_1.

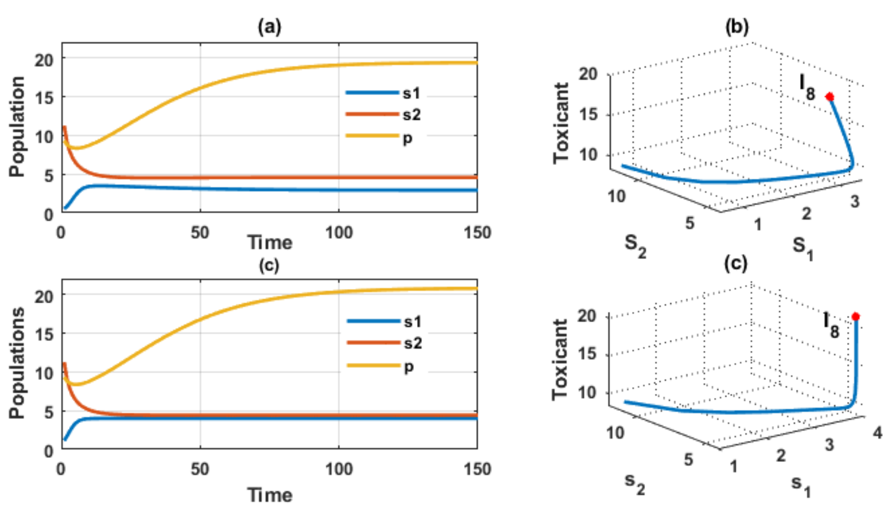

Figure 3 Dynamics of system (1) with (a) time series with $\beta_1 = 0.9$*, system (1) converges to (2.9,4.5,19.3); (b) phase diagram of (a); (c) time series with* $\beta_1 = 0.003$*, system (1) converges (3.9,4.4,20.8); (d) phase diagram of (c).*

To numerically explore the effect of β_2 the parameters in (46) remain the same except for changing β_2. The solution of system (1) asymptotically approaches I_8 for different values of β_2. (See Figure 4).

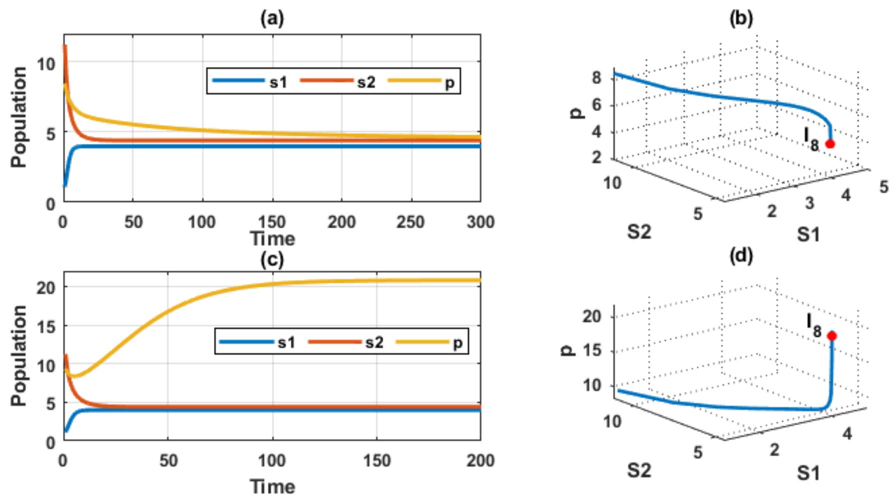

Figure 4 Dynamics of system (1) with (a) time series with $\beta_2 = 0.9$*, system (1) converges to (3.9, 4.4, 4.5); (b) phase diagram of (a); (c) time series with* $\beta_2 = 0.00001$*, system (1) converges (3.9, 4.4,20.8); (d) phase diagram of (c).*

The same scenario can be detected with changing α_1, α_2 and d, The solution of system (1) converges asymptotically to its interior point I_8 for different values of them. (See Figures 5-8).

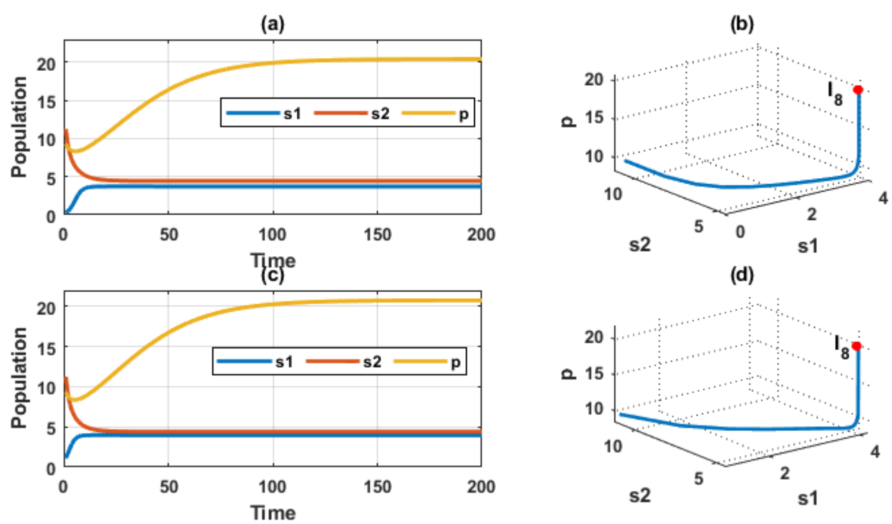

Figure 5 Dynamics of system (1) with (a) time series with $\alpha_1 = 0.9$*, system (1) converges to (3.7, 4.4, 20.4); (b) phase diagram of (a); (c) time series with* $\alpha_1 = 0.00006$*, system (1) converges (3.9, 4.4, 20.8); (d) phase diagram of (c).*

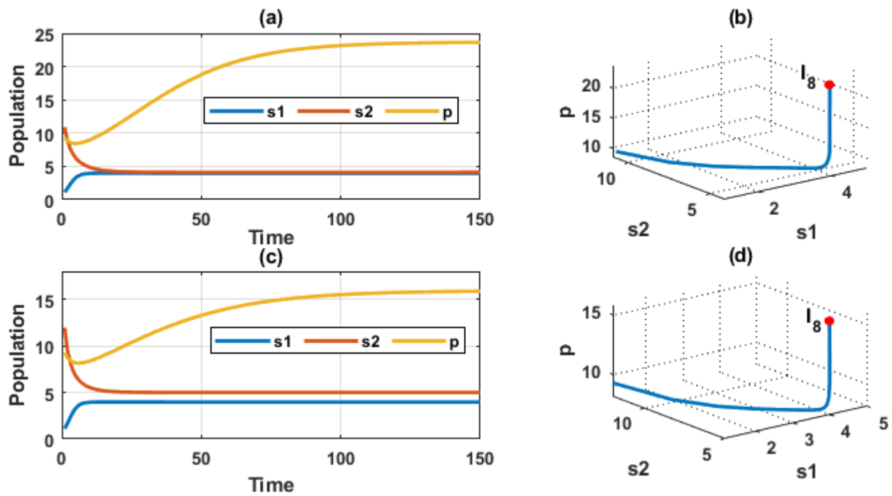

Figure 6 Dynamics of system (1) with (a) time series with $\alpha_2 = 0.9$*, system (1) converges to (3.9, 4.1, 23.6); (b) phase diagram of (a); (c) time series with* $\alpha_2 = 0.0002$*, system (1) converges (3.9,4.9, 15.9); (d) phase diagram of (c).*

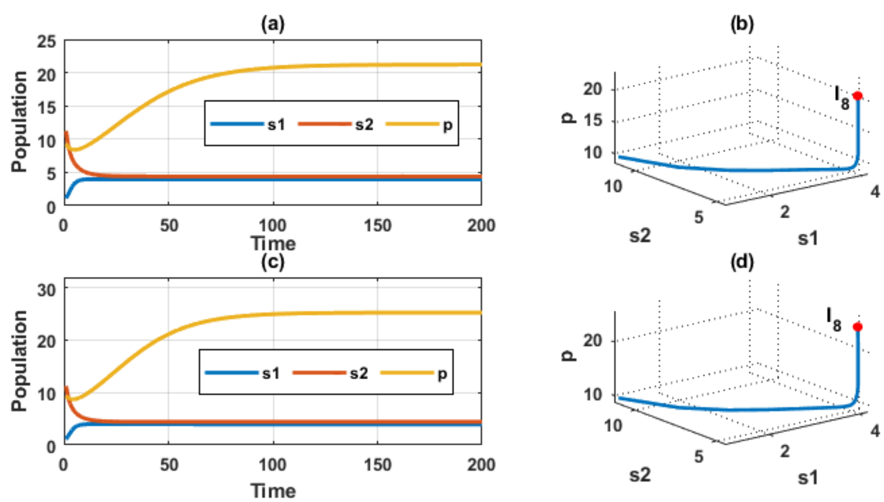

Figure 7 Dynamics of system (1) with (a) time series with $d = 0.9$*, system (1) converges to (3.9, 4.4, 21.2); (b) phase diagram of (a); (c) time series with* $d = 0.003$*, system (1) converges (3.9, 4.4,25.21); (d) phase diagram of (c).*

Now, Figure 8 explains system's (1) dynamics with the data given by (46), with different values of α_3. It illustrates the solution of system (1) stabilising at I_8, when $\alpha_3 > 0.11$. While the solution of system (1) settles down to I_7 in Int.$R^2_{+(s_1 s_2)}$, when $\alpha_3 \leq 0.11$.

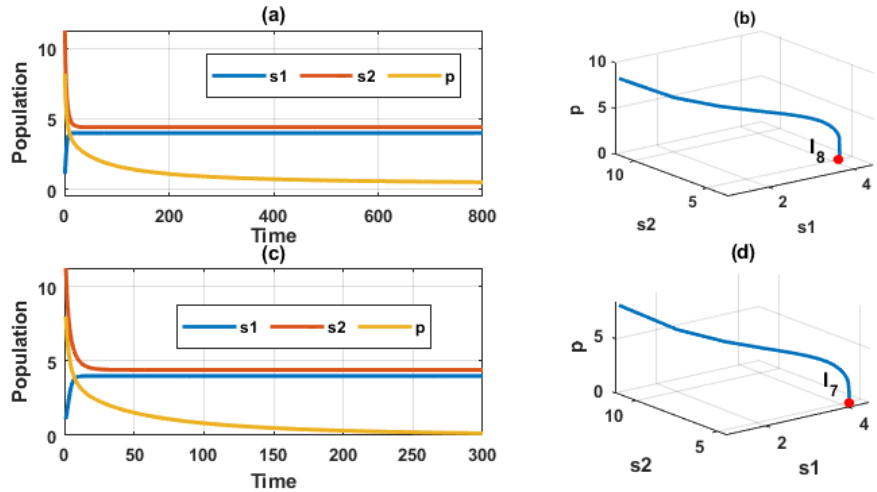

Figure 8 Dynamics of system (1) with (a) time series with $\alpha_3 = 0.1$*, system (1) converges to (3.9, 4.4, 0.4); (b) phase diagram of (a); (c) time series with* $\alpha_3 = 0.11$*, system (1) converges (3.9, 4.4,0); (d) phase diagram of (c).*

Figure 9 illustrates the system (1) dynamics with (46) at various values of r_1. It demonstrates that when $0.6 \leq r_1 \leq 68.7$, the solution of system (1) approaches its positive balance point I_8. Furthermore, for $r_1 < 0.6$ and $r_1 > 68.7$, the first species becomes zero and the solution approach asymptotically to I_5 in Int.$R^2_{+(s_2 p)}$.

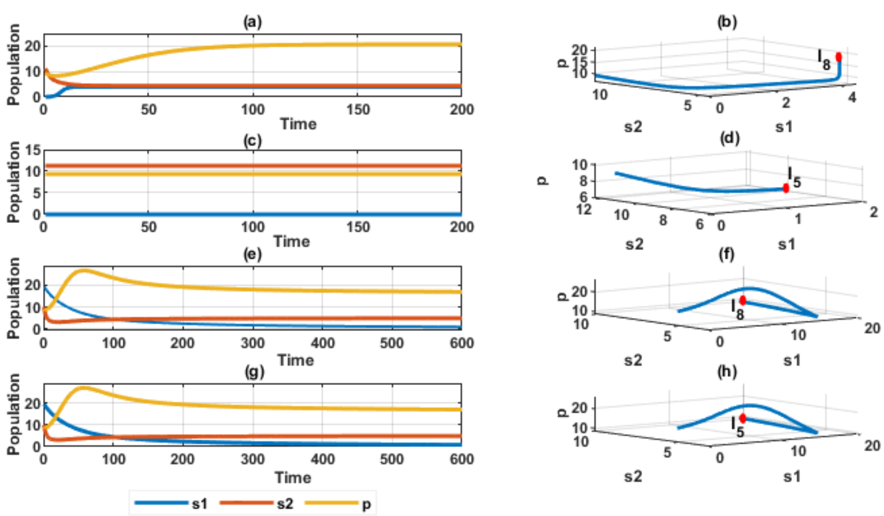

Figure 9 Dynamics of system (1) with (a) time series with $r_1 = 68.7$*, system (1) converges to (3.9, 4.4, 20.7); (b) phase diagram of (a); (c) time series with* $r_1 = 68.8$*, system (1) converges (0, 11.2, 9.3); (d) phase diagram of (c). (e) time series with* $r_1 = 0.6$*, system (1) converges (0.004, 4.9, 16); (f) phase diagram of (e). (g) time series with* $r_1 = 0.59$*, system (1) converges (0, 5, 16); (h) phase diagram of (g).*

Further, Figure 10 presents the effect of varying r_2 on the dynamics of system (1). It shows the solution approaches I_8 when $r_2 > 0.001$. Furthermore, the first species losses persistence when $r_2 \leq 0.001$. For example, when $r_2 = 0.001$ the solution, in this case, approaches to $I_5 = (0, 0.8, 2165)$.

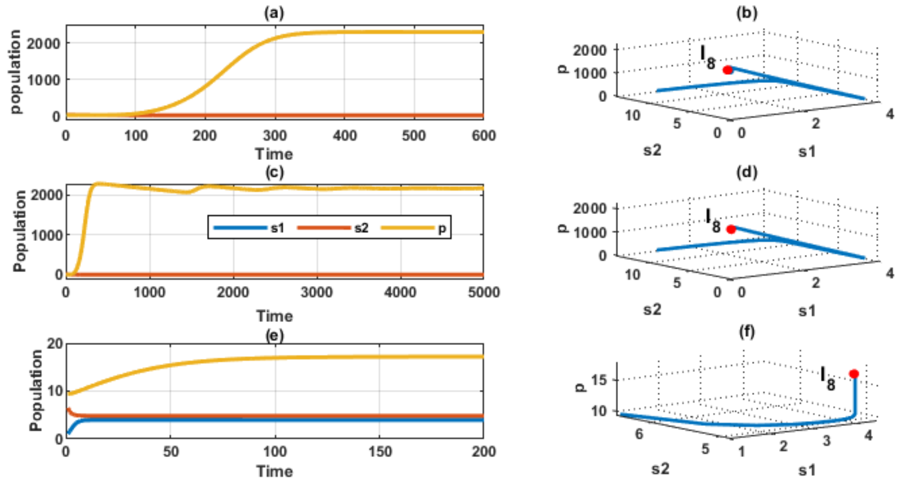

Figure 10 Dynamics of system (1) with (a) time series with $r_2 = 0.01$*, system (1) converges to (0.01, 0.08, 2157); (b) phase diagram of (a); (c) time series with* $r_2 = 0,001$*, system (1) converges (0, 0.8, 2165.4); (d) phase diagram of (c). (e) time series with* $r_2 = 70$*, system (1) converges (3.9, 4.8, 17.1); (f) phase diagram of (e).*

Finally, Figure 11 shows the impact of varying r_3 on the system's (1) behaviour. Clearly, the solution of system (1) accesses its positive equilibrium point I_8 when $r_3 > 4.9$. While the solution of system (1) settles down to I_7 in Int.$R^2_{+(s_1 s_2)}$, when $r_3 \leq 4.9$.

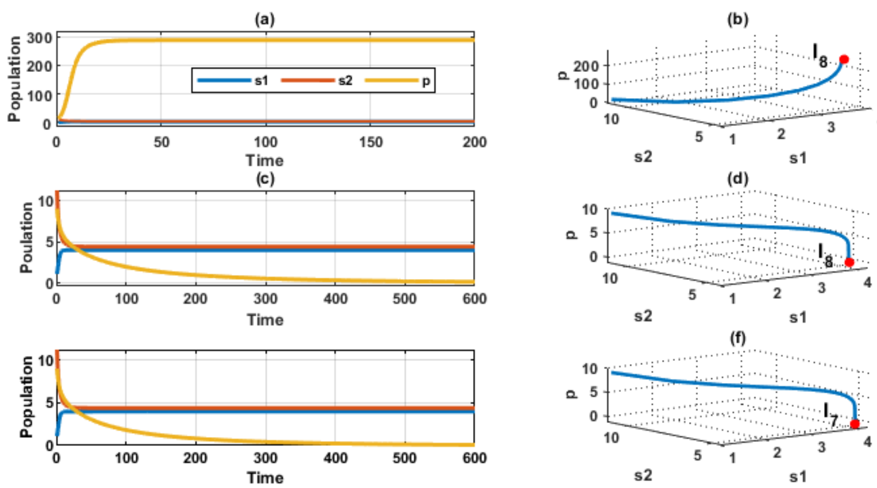

Figure 11 Dynamics of system (1) with (a) time series with $r_3 = 70$*, system (1) converges to (3.4, 4.4, 287.9); (b) phase diagram of (a); (c) time series with* $r_3 = 5$*, system (1) converges (3.9, 4.4, 1.08); (d) phase diagram of (c). (e) time series with* $r_3 = 4.9$*, system (1) converges (3.9, 4.4, 0); (f) phase diagram of (e).*

10. CONCLUSION

A two-competitive species model with pollution has been proposed and intensively studied. The type I functional response has been provided to describe the negative effects of the first species due to the toxins in the environment. The type II functional response has been supposed to represent the toxin's degradation due to the existence of the second species. The theoretical examination shows the existing conditions of the eight non-negative fixed points. Based on the Routh-Hurwitz stability criteria, the local stability of all steadiness points has been studied.

The global dynamics of equilibria have been established by using the Lyapunov method. Further, the Sotomayor theorem has been applied to estimate the appearance of local bifurcation near the equilibrium points. Finally, a 3D phases diagram and time series have been utilised to confirm the analytical result. The result shows that system (1) movement always occurs around the interior steady state if the stability conditions are met. In contrast, a varying in the growth rates (r_1, r_2, r_3) and the uptake rate of toxicants by the second species will lead to the damage of some species.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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