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Topological Generalizations of Rough Concepts

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Abstract

The importance of topology as a tool in preference theory is what motivates this study in which we characterize topologies generating by digraphs. In this paper, we generalized the notions of rough set concepts using two topological structures generated by out (resp. in)-degree sets of vertices on general digraph. New types of topological rough sets are initiated and studied using new types of topological sets. Some properties of topological rough approximations are studied by many propositions.

Key words: Digraph, Out-degree sets, In-degree sets, Topological spaces, Subbase, Rough sets, Rough approximations, Rough measures.

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1. Introduction and Preliminaries

Rough set theory, proposed in [9-14], is a good mathematical tool for data representation. Its methodology is concerned with the classification and analysis of missing attribute values, uncertain or incomplete information systems and knowledge, and it is considered one of the first non-statistical approaches in data analysis [5, 6, 7]. The subset generated by lower approximation is characterized by certain objects that will definitely form part of an interest subset, whereas the upper approximation is characterized by uncertain objects that will possibly form part of an interest subset. Every subset defined through upper and lower approximation is known as rough set.

For a long time, many individuals believed that abstract topological structures have limited application in the generalization of real line and complex plane or some connections to Algebra and other branches of mathematics. And it seems that there is a big gap between these structures and real life applications. We noticed that in some situations, the concept of relation is used to get topologies that are used in important applications such as computing topologies [16], recombination spaces [2, 4, 17] and information granulation which are used in biological sciences and some other fields of applications.

A directed graph or digraph [15] is a pair $G = (V(G), E(G))$ where $V(G)$ is a non-empty set (called vertex set) and $E(G)$ of ordered pairs of elements of $V(G)$ (called edge set). An edge of the from (v, v) is called a loop. If $v \in V(G)$, the out-degree of v is $|\{u \in V(G) : (v, u) \in E(G)\}|$ and in-degree of v is $|\{u \in V(G) : (u, v) \in E(G)\}|$. A digraph is reflexive if $(v, v) \in E(G)$ for each $v \in V(G)$, symmetric if $(v, u) \in E(G)$ implies $(u, v) \in E(G)$, transitive if $(v, u) \in E(G)$ and $(u, w) \in E(G)$ implies $(v, w) \in E(G)$. A subgraph of a graph G is a graph each of whose vertices belong to $V(G)$ and each of whose edges belong to $E(G)$.

Let X be a finite set, the universe of discourse, and R be an equivalence relation on X [5, 10], called an indiscernability relation. The pair $K = (X, R)$ is called a Pawlak's approximation spaces. R generates a partition $X/R = \{R_x : x \in X\}$ on X where R_x are the equivalence classes for $x \in X$ generated by the equivalence relation R. In the rough set theory, these are also called elementary sets of R. Every finite union of elementary sets in K will be called a composed set in K. The family of all composed sets in K will be denoted by $com(K)$. The family $com(K)$ in the approximation space $K = (X, R)$ is a topology on the set X [11]. For any $A \subseteq X$, the lower $L(A)$ approximation and upper $U(A)$ approximation of A are defined as:

$$L(A) = \{x \in X : R_x \subseteq A\} \text{ and } U(A) = \{x \in X : R_x \cap A \neq \phi\}.$$

Boundary, positive and negative regions are also defined:

$$Bd(A) = U(A) - L(A),\ POS(A) = L(A), \text{ and } NEG(A) = X - U(A).$$

The reference space in rough set theory is the approximation space, whose topology (X, $com(K)$) generated by the equivalence classes of R. In this topology, the closure and interior operators are the same of the upper and lower approximation operators. Moreover, this topology belongs to a special class known by Clopen topology, in which every open set is closed and vice versa. Clopen topology is called the quasi-discrete topology [1].

We will express rough set properties in terms of topological concepts. Let A be a subset of X. Let $Cl(A)$, $Int(A)$ and $Bd(A)$ be closure, interior, and boundary points respectively. A is exact if $Bd(A) = \phi$, otherwise A is rough. It is clear A is exact iff $Cl(A) = Int(A)$. In Pawlak space a subset $A \subseteq X$ has two possibilities rough or exact. For a general topological space, $A \subseteq X$ has the following types of definability:

(a) A is totally definable if A is exact set "$Int(A) = A = Cl(A)$",
(b) A is internally definable if $A = Int(A)$, $A \neq Cl(A)$,
(c) A is externally definable if $A \neq Int(A)$, $A = Cl(A)$,
(d) A is undefinable if $A \neq Int(A)$, $A \neq Cl(A)$.

Original rough membership function is defined using equivalence classes. We will extend it to topological spaces. If $\mathfrak{T}$ is a topology on a finite set X, where its base is β, then the rough membership function is

$$\mu_A^{\mathfrak{T}}(x) = \frac{|\{\cap \beta_x\} \cap A|}{|\{\cap \beta_x\}|},\quad \beta_x \in \beta, x \in X$$

where β_x is any member of β containing x. It can be shown that this number is independent of the choice of bases. Since, the intersection of all members of the topology containing A concedes with the intersection of all members of a base containing x.

2. Topologies Generalized by Out(resp. In)-Degree Sets and Rough Topological Approximations

In this section we introduce the basic notations to topological lower and topological upper approximations. Here we define two topologies generated by any digraph. The subbase of the first topology $\mathfrak{T}_{(v)D}$ (out-degree topology) is the out-degree set $(v)D$. Also, the second topology $\mathfrak{T}_{D(v)}$ (in-degree topology) is the in-degree set $D(v)$ where, $(v)D$, $D(v)$, $\mathfrak{T}_{(v)D}$ and $\mathfrak{T}_{D(v)}$ are define as follows:

Definition 2.1. Let $G = (V(G), E(G))$ be a digraph and a vertex $v \in V(G)$.

(a) The out-degree set of v is denoted by $(v)D$ and defined as:
$(v)D = \{u \in V(G) : (v, u) \in E(G)\}$.
(b) The in-degree set of v is denoted by $D(v)$ and defined as:
$D(v) = \{u \in V(G) : (u, v) \in E(G)\}$.

Definition 2.2. Let $G = (V(G), E(G))$ be a digraph, then

(a) The class $\{(v)D : v \in V(G)\}$ is a subbase for the topology $\mathfrak{T}_{(v)D}$ on G.
(b) The class $\{D(v) : v \in V(G)\}$ is a subbase for the topology $\mathfrak{T}_{D(v)}$ on G.

The topological lower and the topological upper approximations of a subgraph $H = (V(H), E(H))$ of G are defined using the topologies $\mathfrak{T}_{(v)D}$ and $\mathfrak{T}_{D(v)}$ as follows:

$L_{\mathfrak{T}(v)D}(H) = \bigcup\{(v)D : (v)D \subseteq V(H)\}$ and

$U_{\mathfrak{T}(v)D}(H) = \bigcup\{(v)D : (v)D \cap V(H) \neq \phi\}$.

$L_{\mathcal{T}D(v)}(H) = \cup\{D(v) : D(v) \subseteq V(H)\}$ and
$U_{\mathcal{T}D(v)}(H) = \cup\{D(v) : D(v) \cap V(H) \neq \phi\}$.

Some types of topological rough sets are initiated in the following definition.

Definition 2.3. Let $\mathcal{G} = (G, D)$ be a generalized approximation space. Let $\mathcal{T}_{(v)D}$ and $\mathcal{T}_{D(v)}$ be the two topologies generated using the relation D. Then the subgraph $H \subseteq G$ is called:

(a) Semi rough (briefly S_{12}-rough) if $V(H) \subseteq U_{\mathcal{T}D(v)}(L_{\mathcal{T}(v)D}(H))$.
(b) Pre rough (briefly P_{12}-rough) if $V(H) \subseteq U_{\mathcal{T}(v)D}(L_{\mathcal{T}D(v)}(H))$.
(c) Semi-pre rough (briefly β_{12}-rough) if $V(H) \subseteq U_{\mathcal{T}D(v)}(L_{\mathcal{T}(v)D}(U_{\mathcal{T}D(v)}(H)))$.
(d) α-Rough (briefly α_{12}-rough) if $V(H) \subseteq L_{\mathcal{T}(v)D}(U_{\mathcal{T}D(v)}(L_{\mathcal{T}(v)D}(H)))$.
(e) γ-Rough (briefly γ_{12}-rough) if $V(H) \subseteq U_{\mathcal{T}D(v)}(L_{\mathcal{T}(v)D}(H)) \cap L_{\mathcal{T}(v)D}(U_{\mathcal{T}D(v)}(H))$.

The family of all S_{12}-rough (resp. P_{12}-rough, β_{12}-rough, α_{12}-rough and γ_{12}-rough) sets in $\mathcal{G} = (G, D)$ is denoted by $FS_{12}(G)$ (resp. $FP_{12}(G)$, $F\beta_{12}(G)$, $F\alpha_{12}(G)$ and $F\gamma_{12}(G)$).

The complement of S_{12}-rough (resp. P_{12}-rough, β_{12}-rough, α_{12}-rough and γ_{12}-rough) set is called S^c_{12}-rough (resp. P^c_{12}-rough, β^c_{12}-rough, α^c_{12}-rough and γ^c_{12}-rough).

The family of all S^c_{12}-rough (resp. P^c_{12}-rough, β^c_{12}-rough, α^c_{12}-rough and γ^c_{12}-rough) sets of $\mathcal{G} = (G, D)$ is denoted FS^c_{12}-rough (resp. FP^c_{12}-rough, $F\beta^c_{12}$-rough, $F\alpha^c_{12}$-rough and $F\gamma^c_{12}$-rough).

Proposition 2.1. In the generalized approximation space $\mathcal{G} = (G, D)$, we can prove that:

(a) $F\alpha_{12}(G) \subseteq FS_{12}(G) \subseteq F\gamma_{12}(G) \subseteq F\beta_{12}(G)$.
(b) $F\alpha_{12}(G) \subseteq FP_{12}(G) \subseteq F\gamma_{12}(G) \subseteq F\beta_{12}(G)$.

Proof. Obvious.

The following example illustrates the above definition.

Example 2.1. Let $G = (V(G), E(G))$ be a digraph such that $V(G) = \{v_1, v_2, v_3, v_4\}$ and $E(G) = \{(v_1, v_1), (v_1, v_3), (v_1, v_4), (v_2, v_2), (v_2, v_4), (v_3, v_1), (v_3, v_2), (v_3, v_4), (v_4, v_1)\}$. Hence the subbase of $\mathcal{T}_{(v)D}$ is $\{\{v_1, v_3, v_4\}, \{v_2, v_4\}, \{v_1, v_2, v_4\}, \{v_1\}\}$ and the subbase of $\mathcal{T}_{D(v)}$ is $\{\{v_1, v_3, v_4\}, \{v_2, v_3\}, \{v_1\}, \{v_1, v_2, v_3\}\}$. Then

$\mathcal{T}_{(v)D} = \{G, \phi, \{v_1, v_3, v_4\}, \{v_2, v_4\}, \{v_1, v_2, v_4\}, \{v_1\}, \{v_4\}, \{v_1, v_4\}\}$,
$\mathcal{T}_{D(v)} = \{G, \phi, \{v_1, v_3, v_4\}, \{v_1, v_3\}, \{v_1, v_2, v_3\}, \{v_1\}, \{v_3\}, \{v_2, v_3\}\}$.

Consequently,
$F\alpha_{12}(G) = FS_{12}(G) = \{G, \phi, \{v_1, v_3, v_4\}, \{v_2, v_4\}, \{v_1, v_2, v_4\}, \{v_1\}, \{v_4\}, \{v_1, v_4\}\}$,
$F\gamma_{12}(G) = F\beta_{12}(G) = \{G, \phi, \{v_1\}, \{v_4\}, \{v_1, v_2\}, \{v_1, v_4\}, \{v_1, v_3\}, \{v_1, v_2, v_3\}, \{v_2, v_4\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}\}$.

Definition 2.4. Let $\mathcal{G} = (G, D)$ be a generalized approximation space and subgraph $H \subseteq G$. Then the general lower (briefly J_{12} lower) of H denoted by $L_{J12}(H)$ for all $J_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$ is defined by: $L_{J12}(H) = \cup\{O \in FJ_{12}(G) : O \subseteq V(H)\}$.

Definition 2.5. Let $\mathcal{G} = (G, D)$ be a generalized approximation space and subgraph $H \subseteq G$. Then the general upper (briefly J_{12} upper) of H denoted by $U_{J12}(H)$ for all $J_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$ is defined by: $U_{J12}(H) = \cap\{O \in FJ^c_{12}(G) : V(H) \subseteq O\}$.

Definition 2.6. Let $\mathcal{G} = (G, D)$ be a generalized approximation space. Then for all $J_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$ the topological general lower and topological general upper approximations of any subgraph $H \subseteq G$ are defined as: $RL_{J12}(H) = L_{J12}(H)$, $RU_{J12}(H) = U_{J12}(H)$.

Proposition 2.2. Let $\mathcal{G} = (G, D)$ be a generalized approximation space generated by any digraph $G = (V(G), E(G))$. Then for any subgraph $H \subseteq G$.

(a) $L_{\mathcal{T}(v)D}(H) \subseteq L_{\alpha 12}(H) \subseteq L_{S12}(H) \subseteq L_{\gamma 12}(H) \subseteq L_{\beta 12}(H) \subseteq V(H) \subseteq U_{\beta 12}(H) \subseteq U_{\gamma 12}(H) \subseteq U_{S12}(H) \subseteq U_{\alpha 12}(H) \subseteq U_{\mathcal{T}(v)D}(H)$.

(b) $L_{\mathcal{T}D(v)}(H) \subseteq L_{\alpha 12}(H) \subseteq L_{P12}(H) \subseteq L_{\gamma 12}(H) \subseteq L_{\beta 12}(H) \subseteq V(H) \subseteq U_{\beta 12}(H) \subseteq U_{\gamma 12}(H) \subseteq U_{P12}(H) \subseteq U_{\alpha 12}(H) \subseteq U_{\mathcal{T}D(v)}(H)$.

Proof.

(a) $L_{\mathcal{T}(v)D}(H) = \cup\{O \in \mathcal{T}_{(v)D} : O \subseteq V(H)\} \subseteq \cup\{O \in F\alpha_{12}(G) : O \subseteq V(H)\}$
$\subseteq \cup\{O \in FS_{12}(G) : O \subseteq V(H)\} \subseteq \cup\{O \in F\gamma_{12}(G) : O \subseteq V(H)\}$
$\subseteq \cup\{O \in F\beta_{12}(G) : O \subseteq V(H)\} \subseteq V(H) \subseteq \cap\{O \in F\beta_{12}^{c}(G) : V(H) \subseteq O\}$
$\subseteq \cap\{O \in F\gamma_{12}^{c}(G) : V(H) \subseteq O\} \subseteq \cap\{O \in FS_{12}^{c}(G) : V(H) \subseteq O\}$
$\subseteq \cap\{O \in F\alpha_{12}^{c}(G) : V(H) \subseteq O\} \subseteq \cap\{O \in \mathcal{T}^{c}_{(v)D} : V(H) \subseteq O\}$.

Hence, $L_{\mathcal{T}(v)D}(H) \subseteq L_{\alpha 12}(H) \subseteq L_{S12}(H) \subseteq L_{\gamma 12}(H) \subseteq L_{\beta 12}(H) \subseteq V(H) \subseteq U_{\beta 12}(H) \subseteq U_{\gamma 12}(H) \subseteq U_{S12}(H) \subseteq U_{\alpha 12}(H) \subseteq U_{\mathcal{T}(v)D}(H)$.

(b) By the same manner as (a).

Example 2.2. According to Example 2.1, if $H = (V(H), E(H))$ and $K = (V(K), E(K))$ be two digraph such that $V(H) = \{v_1, v_2\}$, $V(K) = \{v_4\}$, $E(H) = \{(v_1, v_1), (v_2, v_2)\}$ and $E(K) = \phi$. Then $L_{\alpha 12}(H) = \{v_1\}$, $L_{P12}(H) = \{v_1, v_2\}$, $U_{\alpha 12}(K) = \{v_2, v_3, v_4\}$, and $U_{P12}(K) = \{v_4\}$. So $L_{\alpha 12}(H) \subseteq L_{P12}(H)$ and $U_{P12}(K) \subseteq U_{\alpha 12}(K)$.

Proposition 2.3. Let $\mathcal{G} = (G, D)$ be a generalized approximation space generated by any digraph G. Then for any two subgraphs $H, K \subseteq G$ we have for all $J_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$:

(a) $L_{J12}(\phi) = U_{J12}(\phi) = \phi$, $L_{J12}(G) = U_{J12}(G) = V(G)$.
(b) If $H \subseteq K$, then $L_{J12}(H) \subseteq L_{J12}(K)$.
(c) If $H \subseteq K$, then $U_{J12}(H) \subseteq U_{J12}(K)$.
(d) $L_{J12}(H \cup K) \supseteq L_{J12}(H) \cup L_{J12}(K)$.
(e) $U_{J12}(H \cup K) \supseteq U_{J12}(H) \cup U_{J12}(K)$.
(f) $L_{J12}(H \cap K) \subseteq L_{J12}(H) \cap L_{J12}(K)$.
(g) $U_{J12}(H \cap K) \subseteq U_{J12}(H) \cap U_{J12}(K)$.
(h) $L_{J12}(H^{c}) = [U_{J12}(H)]^{c}$.
(i) $U_{J12}(H^{c}) = [L_{J12}(H)]^{c}$.

Proof. By using the properties of $L_{J12}(H)$ and $U_{J12}(H)$ for all $J_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$ the proof is complete.

The following example, at $J_{12} = \alpha_{12}$ illustrates that the inverse of Property (d) in the above proposition in general does not hold for all $J_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$.

Example 2.3. According to Example 2.1, if $H = (V(H), E(H))$ and $K = (V(K), E(K))$ be two digraph such that $V(H) = \{v_1\}$, $V(K) = \{v_3, v_4\}$, $E(H) = \{(v_1, v_1)\}$ and $E(K) = \{(v_3, v_4)\}$. Then $L_{\alpha 12}(H) = \{v_1\}$, $L_{\alpha 12}(K) = \{v_4\}$, and $L_{\alpha 12}(H \cup K) \neq L_{\alpha 12}(H) \cup L_{\alpha 12}(K)$.

The following example shows that the inverse of the Properties (e) and (f) in Proposition 2.2, in general are not true for all $J_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$, we consider $J_{12} = \beta_{12}$.

Example 2.4. According to Example 2.1, if $H_1 = (V(H_1), E(H_1))$, $H_2 = (V(H_2), E(H_2))$, $K_1 = (V(K_1), E(K_1))$ and $K_2 = (V(K_2), E(K_2))$ be four digraph such that $V(H_1) = \{v_1\}$, $V(H_2) = \{v_4\}$, $V(K_1) = \{v_1, v_2\}$, $V(K_2) = \{v_2, v_4\}$, $E(H_1) = \{(v_1, v_1)\}$, $E(H_2) = \phi$, $E(K_1) = \{(v_1, v_1), (v_2, v_2)\}$, and $E(K_2) = \{(v_2, v_2), (v_2, v_4)\}$. Then $U_{\beta12}(H_1) = \{v_1, v_3\}$, $U_{\beta12}(H_2) = \{v_4\}$, $U_{\beta12}(H_1 \cup H_2) = V(G)$, $L_{\beta12}(K_1) = \{v_1, v_2\}$, $L_{\beta12}(K_2) = \{v_2, v_4\}$, and $L_{\beta12}(K_1 \cap K_2) = \phi$. Hence $U_{\beta12}(H_1 \cup H_2) \neq U_{\beta12}(H_1) \cup U_{\beta12}(H_2)$ and $L_{\beta12}(K_1 \cap K_2) \neq L_{\beta12}(K_1) \cap L_{\beta12}(K_2)$.

The following example shows that the Property (g) in Proposition 2.2, in general are not true for all $J_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$, we consider $J_{12} = P_{12}$.

Example 2.5. According to Example 2.1, if $H = (V(H), E(H))$ and $K = (V(K), E(K))$ be two digraph such that $V(H) = \{v_1, v_3, v_4\}$, $V(K) = \{v_2, v_3, v_4\}$, $E(H) = \{(v_1, v_1), (v_1, v_3), (v_1, v_4), (v_3, v_1), (v_3, v_4), (v_4, v_1)\}$ and $E(K) = \{(v_2, v_2), (v_2, v_4), (v_3, v_2), (v_3, v_4)\}$. Then $U_{P12}(H) = V(G)$, $U_{P12}(K) = \{v_2, v_3, v_4\}$ and $U_{P12}(H \cap K) = \{v_3, v_4\}$. Hence $U_{P12}(H \cap K) \neq U_{P12}(H) \cap U_{P12}(K)$.

Remark 2.1. Let $\mathcal{G} = (G, D)$ be a generalized approximation space defined on any digraph G. Then for all $J_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$, and for any subgraph $H \subseteq G$, the following properties do not hold:

(a) $L_{J12}(L_{J12}(H)) = U_{J12}(L_{J12}(H)) = L_{J12}(H)$.

(b) $U_{J12}(U_{J12}(H)) = L_{J12}(U_{J12}(H)) = U_{J12}(H)$.

The following example illustrates the above remark, using $J_{12} = \beta_{12}$.

Example 2.6. According to Example 2.1, if $H = (V(H), E(H))$ and $K = (V(K), E(K))$ be two digraph such that $V(H) = \{v_1\}$, $V(K) = \{v_3, v_4\}$, $E(H) = \{(v_1, v_1)\}$ and $E(K) = \{(v_3, v_4)\}$. Then $L_{\beta12}(H) = \{v_1\}$, $L_{\beta12}(L_{\beta12}(H)) = \{v_1\}$, $U_{\beta12}(L_{\beta12}(H)) = \{v_1, v_3\}$, $U_{\beta12}(K) = \{v_3, v_4\}$, $U_{\beta12}(U_{\beta12}(K)) = \{v_3, v_4\}$, $L_{\beta12}(U_{\beta12}(K)) = \{v_4\}$. Hence $L_{J12}(L_{J12}(H)) \neq U_{J12}(L_{J12}(H))$, $U_{J12}(U_{J12}(H)) \neq L_{J12}(U_{J12}(H))$.

Lemma 2.1. Let $\mathcal{G} = (G, D)$ be a generalized approximation space, and for any subgraph $H \subseteq G$, then $[Cl_{J12}(H)]^c = Int_{J12}(H^c)$ for all $J_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$.

Proof. Let $H \subseteq G$, then for all $J_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$, we get:

$[Cl_{J12}(H)]^c = V(G) - Cl_{J12}(H)$

$= V(G) - \cap\{V(F) \subseteq V(G) : F$ is J_{12} upper graph and $V(H) \subseteq V(F)\}$.

$= \cup\{[V(G) - V(F)] \subseteq V(G) : (G - F)$ is J_{12} lower graph and $[V(G) - V(F)] \subseteq [V(G) - V(F)]\} = Int_{J12}(G - H)$.

Thus $[Cl_{J12}(H)]^c = Int_{J12}(H^c)$.

Proposition 2.4. Let $\mathcal{G} = (G, D)$ be a generalized approximation space defined on any digraph G, for any two subgraphs $H, K \subseteq G$ we have: $L_{J12}(H - K) \subseteq L_{J12}(H) - L_{J12}(K)$, for all $J_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$.

Proof. As $H - K = H \cap K^c$, then:

$L_{J12}(H - K) = Int_{J12}(H - K) = Int_{J12}(H \cap K^c) \subseteq Int_{J12}(H) \cap Int_{J12}(K^c)$. By lemma 2.1, we have $L_{J12}(H - K) \subseteq Int_{J12}(H) \cap [Cl_{J12}(K)]^c = Int_{J12}(H) - Cl_{J12}(K) \subseteq Int_{J12}(H) - Int_{J12}(K)$. Thus $L_{J12}(H - K) \subseteq L_{J12}(H) - L_{J12}(K)$.

The next example illustrates that the inverse of Proposition 2.4, in general does not hold with respect to $J_{12} = \beta_{12}$.

Example 2.7. According to Example 2.1, if $H = (V(H), E(H))$ and $K = (V(K), E(K))$ be two digraph such that $V(H) = \{v_1, v_2\}$, $V(K) = \{v_1\}$, $E(H) = \{(v_1, v_1), (v_2, v_2)\}$ and $E(K) = \{(v_1, v_1)\}$. Then $L_{\beta 12}(H) = \{v_1, v_2\}$, $L_{\beta 12}(K) = \{v_1\}$ and $L_{\beta 12}(H - K) = \phi$, thus $L_{J12}(H - K) \neq L_{J12}(H) - L_{J12}(K)$.

3. Topological Generalizations of Rough Concepts

In this section we introduce and study some topological generalizations for some concepts of the rough set theory by using the J_{12} lower and J_{12} upper approximations.

Definition 3.1. Let $\mathcal{G} = (G, D)$ be a generalized approximation space defined on any digraph G. Then for all $J_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$ and for any subgraph $H \subseteq G$ we define:

(a) H is totally topological J_{12}-definable (J_{12}-exact) graph if $L_{J12}(H) = U_{J12}(H) = V(H)$.

(b) H is internally topological J_{12}-definable graph if $L_{J12}(H)=V(H)$ and $U_{J12}(H)\neq V(H)$.

(c) H is externally topological J_{12}-definable graph if $L_{J12}(H)\neq V(H)$ and $U_{J12}(H)=V(H)$.

(d) H is topologically J_{12}-indefinable (J_{12}-rough) graph if $L_{J12}(H)\neq V(H)$ and $U_{J12}(H)\neq V(H)$.

Example 3.1. According to Example 2.1, for subgraphs $H = (V(H), E(H))$ and $K = (V(K), E(K))$ such that $V(H) = \{v_4\}$, $V(K) = \{v_3, v_4\}$, $E(H) = \phi$ and $E(K) = \{(v_3, v_4)\}$, H is topologically β_{12}-exact graph, H is topologically internally α_{12}-definable graph, K is topologically S_{12}-rough graph, and K is topologically externally P_{12}-definable graph.

Definition 3.2. Let $\mathcal{G} = (G, D)$ be a generalized approximation space defined on any digraph G. Then we can introduce the generalized accuracy measure for any graph $H \subseteq G$ as the following:

$$\eta_{J12}(V(H)) = \frac{|L_{J12}(V(H))|}{|U_{J12}(V(H))|},\ U_{J12}(H) \neq \phi,$$

where $J_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$, and $|H|$ denoted the cardinality of the vertex set of H.

The number η_{J12} of the above definition is a measure of the degree of exactness of any subgraph $H \subseteq G$. So by this measure we will determine, what is the best of our definitions for the J_{12} lower and J_{12} upper approximations. We can notice that:

(a) $0 \leq \eta_{\mathcal{T}(v)D}(V(H)) \leq \eta_{\alpha 12}(V(H)) \leq \eta_{S12}(V(H)) \leq \eta_{\gamma 12}(V(H)) \leq \eta_{\beta 12}(V(H)) \leq 1$.

(b) $0 \leq \eta_{\mathcal{T}(v)D}(V(H)) \leq \eta_{\alpha 12}(V(H)) \leq \eta_{P12}(V(H)) \leq \eta_{\gamma 12}(V(H)) \leq \eta_{\beta 12}(V(H)) \leq 1$.

So the best definition here is $J_{12} = \beta_{12}$.

The next example studies the comparison between β_{12} and S_{12}.

Example 3.2. According to Example 2.1, we have the following table:

Subgraph H	$\eta_{S12}(V(H))$	$\eta_{\beta 12}(V(H))$
$\{v_4\}$	1/3	1
$\{v_1, v_3\}$	1/2	1
$\{v_2, v_4\}$	2/3	1
$\{v_1, v_2, v_3\}$	1/3	1

By using the definitions of rough concepts at $J_{12} = \beta_{12}$ we can tends to exactness of many graphs. This will lead to accurate results in many data reduction applications using new topological approaches. Next works shall deal with more types of applications in data reductions, data processing, image processing and rule extraction.

Definition 3.3. Let $\mathcal{G} = (G, D)$ be a generalized approximation space defined on any digraph G. Then for all $J_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$ and for any $H, K \subseteq G$ we call:

(a) H is roughly bottom-part of K in G iff $L_{J12}(H) \subseteq L_{J12}(K)$ and denoted by $H \underset{\approx}{\subset}_{J12} K$.

(b) H is roughly top-part of K in G iff $U_{J12}(H) \subseteq U_{J12}(K)$ and denoted by $H \tilde{\subset}_{J12} K$.

The illustration of the facts of the above definition are given as below example.

Example 3.3. According to Example 2.1, if $H_1 = (V(H_1), E(H_1))$, $H_2 = (V(H_2), E(H_2))$, $H_3 = (V(H_3), E(H_3))$ and $H_4 = (V(H_4), E(H_4))$ be four digraph such that $V(H_1) = \{v_1, v_3, v_4\}$, $V(H_2) = \{v_2, v_3, v_4\}$, $V(H_3) = \{v_2, v_4\}$, $V(H_4) = \{v_3, v_4\}$, $E(H_1) = \{(v_1, v_1), (v_1, v_3), (v_1, v_4), (v_3, v_1), (v_3, v_4), (v_4, v_1)\}$, $E(H_2) = \{(v_2, v_2), (v_2, v_4), (v_3, v_2), (v_3, v_4)\}$, $E(H_3) = \{(v_2, v_2), (v_2, v_4)\}$, and $E(H_4) = \{(v_3, v_4)\}$. Then we have: $H_4 \underset{\approx}{\subset}_{P12} H_3$, $H_2 \tilde{\subset}_{S12} H_1$ and $H_2 \tilde{\subset}_{\beta12} H_1$.

Definition 3.4. Let $\mathcal{G} = (G, D)$ be a generalized approximation space defined on any digraph G. For any subgraph $H \subseteq G$ and any vertex $v \in V(G)$, for all $J_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$ we call:

(a) v is surely belong in H iff $v \in L_{J12}(H)$ and denoted by $v \underset{\approx}{\in}_{J12} H$.

(b) v is possibly belong in H iff $v \in U_{J12}(H)$ and denoted by $v \tilde{\in}_{J12} H$.

Proposition 3.1. Let $\mathcal{G} = (G, D)$ be a generalized approximation space defined on any digraph G. For any subgraph $H \subseteq G$ and any vertex $v \in V(G)$, for all $J_{12} \in \{S_{12}, P_{12}, \beta_{12}, \alpha_{12}, \gamma_{12}\}$, we have:

(a) if $v \underset{\approx}{\in}_{J12} H$ then $v \in V(H)$.

(b) if $v \tilde{\notin}_{J12} H$ then $v \notin V(H)$.

Proof. The proof is direct from definitions.

The following example shows that the inverse of Proposition 3.1, in general does not hold.

Example 3.4. According to Example 2.1, let $H = (V(H), E(H))$, $K = (V(K), E(K))$ be two digraph such that $V(H) = \{v_1, v_2, v_3\}$, $V(K) = \{v_1, v_4\}$, $E(H) = \{(v_1, v_1), (v_1, v_3), (v_2, v_2), (v_3, v_1), (v_3, v_2)\}$ and $E(K) = \{(v_1, v_1), (v_1, v_4), (v_4, v_1)\}$. Then we have: $v_2 \in V(H)$, but $v_2 \tilde{\notin}_{S12} H$ and $v_2 \tilde{\notin}_{\alpha12} H$. Also, we have $v_2 \notin V(K)$, but $v_2 \tilde{\notin}_{S12} K$, $v_2 \tilde{\notin}_{P12} K$, $v_2 \tilde{\notin}_{\gamma12} K$ and $v_2 \tilde{\notin}_{\beta12} K$.

4. Conclusions

One of the main contributions of this paper is in the area of topological classifications. Based on topological space, we presented an underlying theory to explain how classifications of rough sets topologically may be performed.

We conclude that the intermingling of topology in the construction of some approximation space concepts will help to get results with abundant logical statements. That is discovering hidden relationships among data and, moreover, probably helps in producing accurate programs (Duntsch et al., 2001[3]; Lipski, 1981[8]).

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