A single server fuzzy queues with priority and unequal service rates

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Abstract: This article presents to develop a new mathematical model to construct membership functions of the fuzzy single channel queueing model $F(M_1, M_2)/F(M_1, M_2)/1/PR\text{-}NP$ with two class of non-preemptive priority with unequal service rates. The main idea of this article is to merge between mathematical programming technique and Yager's ranking method by using α-cut approach and Zadeh extension principle to transform fuzzy queues into a family of conventional crisp (distinct) queues in this context. A numerical example is given to explain the validity of this new approach to obtain real exact values under different levels of α in the system.

Keywords: fuzzy queues; two class priority; mathematical programming technique; unequal service rates.

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1 Introduction

The concept of priority queues is encountered in various service applications. Generally, priority queues are divided into two classes; high-priority and low priority, where the customers with high priority are ahead of line to be served before the customers with low priority which is likewise known as non-preemptive priority. A large number of single queue models with non-preemptive priorities have been used as equal mean service rates for all customers. However, some other single channel queues tend to display a different pattern where the mean service rates are not necessarily equal because of the nature of system. Within the context of traditional queueing theory, the inter arrival times and service times are required to follow certain distributions. However, in many practical applications, the statistical information may be obtained more subjectively; i.e., the arrival pattern and service pattern are more suitably described by linguistic terms such as fast, slow, or moderate, using probability distributions. Thus, fuzzy queues are much more realistic than the commonly used crisp queues. A lot of studies have adopted fuzzy queues in their models, such as the work presented by Prade (1980) and Lee and Li (1989) where the analytical results for two types of fuzzy queues was obtained by adopting Zadeh's extension principle. Also, a α-cut procedure was proposed by Negi and Lee (1992) for analysing the fuzzy queue for two variables using simulation, while Chiang Kao and Li (1999) also presented a novel approach for obtaining the membership function of a fuzzy queue system.

Other authors who also ventured into this area of research include Jeeva and Rathnakumari (2012) whose work analysed a batch arrival single server queue with fuzzy vacation and fuzzy parameters and Devaraj and Jayalakshmi (2012) and Devaraj (2012) who also adopted the same technique, although their work placed importance on equal service rate priority queues with the assumption that the service and inter arrival times follow specific probability distributions. More recent literature include Palpandi and Geetharamani (2013), using robust ranking technique for the reduction of fuzzy queues into crisp queues with three priority class and Ramesh and Ghuru (2014) proposed to generate possibility distribution of the fuzzy queue with non-preemptive priority. Generally, previous research on fuzzy queueing models is focused on ordinary queues with one or two fuzzy variables. Hence, in this paper we develop an approach that provides system characteristics for the priority queues with four fuzzy variables. Through α-cuts we transform the fuzzy priority queues to a family of crisp priority queues with the help of nonlinear programming (NLP) solutions, completely and successfully yielding to build the membership functions of the system characteristics. Although, an explicit closed-form expression for the membership function is very difficult to obtain in the case of four fuzzy variables, we develop a characterisation that yields closed-form expressions when interval limits are invertible. Since the system characteristics are described by membership function, the values conserve completely all of fuzziness of arrival rate, service rate and retrial rate. However, the managers or practitioners would prefer one crisp value for one of system characteristics rather than a fuzzy set. In order to overcome this problem, we defuzzify the fuzzy values of system characteristics by Yager's ranking index method.

This section has presented an overview of previous literature narrowing down their different contributions to the aim of this paper. The others sections of this paper will discuss the following: Section 2 will describe briefly the fuzzy priority single channel fuzzy model with non-preemptive, Section 3 will present the adoption of a new mathematical model is presented using the trapezoidal membership function, Section 4 will have illustrate the application of the new mathematical model using a practical numerical example, Sections 5 and 6 will discuss the results obtained, and conclusions.

2 Fuzzy priority queues

This section considers a fuzzy queueing system in which the server operates under two class of non-preemptive priority. The so-called non-preemptive priority of two class mean that if there is no interruption of server, the first class customer just goes to the head of the queue to wait its turn.

Suppose that the approximate arrival and service rates are all known. We assume that customer's arrival rates into each priority are according to Poisson arrival rates where λ_1, is categorised as high priority and λ_2, as low priority. Also, the service rates of these two priorities are exponential service rates as in different means, represented by μ_1 and μ_2 for high and low priority respectively. We refer to this as two class priority as a F(M$_1$, M$_2$)/F(M$_1$, M$_2$)/1/PR-NP queue, where the first F(M$_1$, M$_2$) symbol's denotes the fuzzified exponential inter arrival time, the second F(M$_1$, M$_2$) symbol's denotes the fuzzified exponential service time under non-preemptive priority noting that the system capacity and population size is infinite.

Let $\mu_{\widetilde{\lambda_1}}(w), \mu_{\widetilde{\lambda_2}}(x), \mu_{\widetilde{\mu_1}}(y)$ and $\mu_{\widetilde{\mu_2}}(z)$ be the membership functions of the fuzzy sets represented as $\widetilde{\lambda_1}, \widetilde{\lambda_2}, \widetilde{\mu_1}$ and $\widetilde{\mu_2}$ respectively. Thus the fuzzy sets $\widetilde{\lambda_1}, \widetilde{\lambda_2}, \widetilde{\mu_1}$ and $\widetilde{\mu_2}$ are given as:

$$\widetilde{\lambda_1} = \left\{ \left(w, \mu_{\widetilde{\lambda_1}}(w) \right) / w \in W \right\}, \tag{1}$$

$$\widetilde{\lambda_2} = \left\{ \left(x, \mu_{\widetilde{\lambda_2}}(x) \right) / x \in X \right\}, \tag{2}$$

$$\widetilde{\mu_1} = \left\{ \left(y, \mu_{\widetilde{\mu_1}}(y) \right) / y \in Y \right\}, \tag{3}$$

and

$$\widetilde{\mu_2} = \left\{ \left(z, \mu_{\widetilde{\mu_2}}(z) \right) / z \in Z \right\}. \tag{4}$$

where W, X, Y and Z are the crisp universal sets of the arrival and service rates, respectively. In this study, the system characteristics of interest are the expected number of customers in the queue, which is denoted by $L_{q1(w,x,y,z)}$. Obviously, $L_{q1(\widetilde{\lambda_1},\widetilde{\lambda_2},\widetilde{\mu_1},\widetilde{\mu_2})}$ is a fuzzy number when $\widetilde{\lambda_1}$, $\widetilde{\lambda_2}$, $\widetilde{\mu_1}$ and $\widetilde{\mu_2}$ are all fuzzy number, based on Zadeh's extension principle (Zadeh, 1978). Then, we have the membership function of the expected number of customers in the queue as follows:

$$\mu_{L_{q1(\widetilde{\lambda_1},\widetilde{\lambda_2},\widetilde{\mu_1},\widetilde{\mu_2})}}(z) = \mu_{\widetilde{L_{q1}}}(z) = \sup_{\substack{w\in W\\ x\in X\\ y\in Y\\ z\in Z}} \left\{ \min \left\{ \begin{matrix} \mu_{\widetilde{\lambda_1}}(w), \mu_{\widetilde{\lambda_2}}(x), \mu_{\widetilde{\mu_1}}(y), \mu_{\widetilde{\mu_2}}(z) / z \\ z = L_{q1}(w, x, y, z) \end{matrix} \right\} \right\}. \tag{5}$$

and

$$\mu_{L_{q2(\widetilde{\lambda_1},\widetilde{\lambda_2},\widetilde{\mu_1},\widetilde{\mu_2})}}(z) = \mu_{\widetilde{L_{q2}}}(z) = \sup_{\substack{w\in W\\ x\in X\\ y\in Y\\ z\in Z}} \left\{ \min \left\{ \begin{matrix} \mu_{\widetilde{\lambda_1}}(w), \mu_{\widetilde{\lambda_2}}(x), \mu_{\widetilde{\mu_1}}(y), \mu_{\widetilde{\mu_2}}(z) / z \\ z = L_{q2}(w, x, y, z) \end{matrix} \right\} \right\}. \tag{6}$$

From the results of Gross and Harris (1998) and Morse (1958), we have the expected number of customer in the crisp queue of two class priority (M_1, M_2)/(M_1, M_2)/1/PR-NP queueing system as:

$$Lq^{(1)} = \frac{\lambda_1 \left(\rho_1 / \mu_1 + \rho_2 / \mu_2\right)}{\left(1-\rho_1\right)}, \tag{7}$$

$$Lq^{(2)} = \frac{\lambda_2 \left(\rho_1 / \mu_1 + \rho_2 / \mu_2\right)}{\left(1-\rho_1\right)(1-\rho)}. \tag{8}$$

where

$$\rho_1 \equiv \frac{\lambda_1}{\mu_1}, \rho_2 \equiv \frac{\lambda_2}{\mu_2}, \tag{9}$$

and

$$\rho = \rho_1 + \rho_2 < 1. \tag{10}$$

The equations (7) and (8) are expressed in a complex pattern. Thus, it is difficult to infer the shape of the membership function $\mu_{\widetilde{L_{q1}}}(z)$ and $\mu_{\widetilde{L_{q2}}}(z)$. Therefore, we develop a mathematical programming technique to find the α-cut of $L_{q1(\widetilde{\lambda_1},\widetilde{\lambda_2},\widetilde{\mu_1},\widetilde{\mu_2})}$ and $L_{q2(\widetilde{\lambda_1},\widetilde{\lambda_2},\widetilde{\mu_1},\widetilde{\mu_2})}$ on the basis of the extension principle in the next section.

3 Parametric NLP approach

In this section, we employ a parametric NLP approach to construct the membership function $\mu_{\widetilde{L_{q1}}}(z)$ and $\mu_{\widetilde{L_{q2}}}(z)$ to identify its shape. Zadeh's extension principle which relies on α-cuts of $\widetilde{L_{q1}}$ and $\widetilde{L_{q2}}$, for the α-cuts of $\widetilde{\lambda_1}, \widetilde{\lambda_2}, \widetilde{\mu_1}$ and $\widetilde{\mu_2}$ as crisp intervals as:

$$\lambda_1(\alpha) = \left\{ w \in W \mid \mu_{\widetilde{\lambda_1}}(x) \geq \alpha \right\}, \tag{11}$$

$$\lambda_2(\alpha) = \left\{ x \in X \mid \mu_{\widetilde{\lambda_1}}(x) \geq \alpha \right\}, \tag{12}$$

$$\mu_1(\alpha) = \left\{ y \in Y \mid \mu_{\widetilde{\lambda_1}}(y) \geq \alpha \right\}, \tag{13}$$

and

$$\mu_2(\alpha) = \left\{ z \in Z \mid \mu_{\widetilde{\lambda_2}}(z) \geq \alpha \right\}, \tag{14}$$

The fuzzy arrival rates $\widetilde{\lambda_1}, \widetilde{\lambda_2}$ and fuzzy service rates $\widetilde{\mu_1}, \widetilde{\mu_2}$ with two class priority are all fuzzy numbers. Thus, the α-cuts of $\widetilde{\lambda_1}, \widetilde{\lambda_2}, \widetilde{\mu_1}$ and $\widetilde{\mu_2}$ defined in equations (11)–(14) are crisp intervals with the following forms:

$$\lambda_1(\alpha) = \left[\min_{w \in W} \left\{ w \mid \mu_{\widetilde{\lambda_1}}(w) \geq \alpha \right\}, \max_{w \in W} \left\{ w \mid \mu_{\widetilde{\lambda_1}}(w) \geq \alpha \right\} \right] = \left[w_\alpha^L, w_\alpha^U \right], \tag{15}$$

$$\lambda_2(\alpha) = \left[\min_{x \in X} \left\{ x \mid \mu_{\widetilde{\lambda_2}}(x) \geq \alpha \right\}, \max_{x \in X} \left\{ x \mid \mu_{\widetilde{\lambda_2}}(x) \geq \alpha \right\} \right] = \left[x_\alpha^L, x_\alpha^U \right], \tag{16}$$

$$\widetilde{\mu_1}(\alpha) = \left[\min_{y \in Y} \left\{ y \mid \mu_{\widetilde{\mu_1}}(y) \geq \alpha \right\}, \max_{y \in Y} \left\{ y \mid \mu_{\widetilde{\mu_1}}(y) \geq \alpha \right\} \right] = \left[y_\alpha^L, y_\alpha^U \right], \tag{17}$$

and

$$\widetilde{\mu_2}(\alpha) = \left[\min_{z \in Z} \left\{ z \mid \mu_{\widetilde{\mu_2}}(z) \geq \alpha \right\}, \max_{z \in Z} \left\{ z \mid \mu_{\widetilde{\mu_2}}(z) \geq \alpha \right\} \right] = \left[z_\alpha^L, z_\alpha^U \right], \tag{18}$$

According to equations (15-18), the arrival rates and service rates can be represented by different α levels of possibility. According to the convexity of a fuzzy number (Zimmermann, 2011), the upper and lower bound of $\widetilde{\lambda_1}, \widetilde{\lambda_2}, \widetilde{\mu_1}$ and $\widetilde{\mu_2}$ are functions of α, which can be represented respectively, as follows: $w_\alpha^L = \min \mu_{\widetilde{\lambda_1}}^{-1}(\alpha)$, $w_\alpha^U = \max \mu_{\widetilde{\lambda_1}}^{-1}(\alpha)$, $x_\alpha^L = \min \mu_{\widetilde{\lambda_2}}^{-1}(\alpha)$, $x_\alpha^U = \max \mu_{\widetilde{\lambda_2}}^{-1}(\alpha)$, $y_\alpha^L = \min \mu_{\widetilde{\lambda_1}}^{-1}(\alpha)$, $y_\alpha^U = \max \mu_{\widetilde{\lambda_1}}^{-1}(\alpha)$, $z_\alpha^L = \min \mu_{\widetilde{\lambda_2}}^{-1}(\alpha)$, and $z_\alpha^U = \max \mu_{\widetilde{\lambda_2}}^{-1}(\alpha)$. Consequently, to find the membership functions $\mu_{\widetilde{L_{q1}}}(z)$ and $\mu_{\widetilde{L_{q2}}}(z)$, we can apply the α-cuts approach because the membership functions of $\mu_{\widetilde{L_{q1}}}(z)$ and $\mu_{\widetilde{L_{q2}}}(z)$ which is defined in equations (5) and (6) is characterised by α, by using Zadeh's extension principle. $\mu_{\widetilde{L_{q1}}}(z)$ and $\mu_{\widetilde{L_{q2}}}(z)$ is the minimum of $\mu_{\widetilde{\lambda_1}}(w)$, $\mu_{\widetilde{\lambda_2}}(x)$, $\mu_{\widetilde{\mu_1}}(y)$ and $\mu_{\widetilde{\mu_2}}(z)$. From these equations, we need that at least one of the following four cases holds:

- Case 1: $\mu_{\tilde{\lambda}_1}(w) = \alpha, \mu_{\tilde{\lambda}_2}(x) \ge \alpha, \mu_{\tilde{\mu}_1}(y) \ge \alpha, \mu_{\tilde{\mu}_2}(z) \ge \alpha$
- Case 2: $\mu_{\tilde{\lambda}_1}(w) \ge \alpha, \mu_{\tilde{\lambda}_2}(x) = \alpha, \mu_{\tilde{\mu}_1}(y) \ge \alpha, \mu_{\tilde{\mu}_2}(z) \ge \alpha$
- Case 3: $\mu_{\tilde{\lambda}_1}(w) \ge \alpha, \mu_{\tilde{\lambda}_2}(x) \ge \alpha, \mu_{\tilde{\mu}_1}(y) = \alpha, \mu_{\tilde{\mu}_2}(z) \ge \alpha$
- Case 4: $\mu_{\tilde{\lambda}_1}(w) \ge \alpha, \mu_{\tilde{\lambda}_2}(x) \ge \alpha, \mu_{\tilde{\mu}_1}(y) \ge \alpha, \mu_{\tilde{\mu}_2}(z) = \alpha$.

Such that

$$z = \frac{\lambda_1\left(\rho_1 / \mu_1 + \rho_2 / \mu_2\right)}{\left(1-\rho_1\right)}, \tag{19}$$

and

$$z = \frac{\lambda_2\left(\rho_1 / \mu_1 + \rho_2 / \mu_2\right)}{\left(1-\rho_1\right)(1-\rho)}. \tag{20}$$

To satisfy $\mu_{\widetilde{L_{q1}}}(z) = \alpha$ and $\mu_{\widetilde{L_{q2}}}(z) = \alpha$, where ρ_1, ρ_2 and ρ are given in equations (9) and (10), respectively. This can be done by means of the parametric NLP technique. For Case 1, we have the lower and upper bound of the α-cut of $\mu_{\widetilde{L_{q1}}}(z)$ and $\mu_{\widetilde{L_{q2}}}(z)$ obtained via this technique as follows:

$$\left(\widetilde{L_{q1}}\right)_\alpha^{LB_1} = w, x, y, z \in R^{*Min} \frac{w\left(\frac{x}{y} / y + \frac{x}{z} / z\right)}{\left(1-\frac{w}{y}\right)}, \tag{21}$$

s.t. $w_\alpha^L \le w \le w_\alpha^U$, $x \in \lambda_2(\alpha)$, $y \in \mu_1(\alpha)$, $z \in \mu_2(\alpha)$

and

$$\left(\widetilde{L_{q1}}\right)_\alpha^{UB_1} = w, x, y, z \in R^{*Max} \frac{w\left(\frac{x}{y} / y + \frac{x}{z} / z\right)}{\left(1-\frac{w}{y}\right)}, \tag{22}$$

s.t. $w_\alpha^L \le w \le w_\alpha^U$, $x \in \lambda_2(\alpha)$, $y \in \mu_1(\alpha)$, $z \in \mu_2(\alpha)$.

Then we use the same procedure to obtain the lower and upper bounds for Cases 2, 3 and 4 respectively, in the following manner:

$$\left(\widetilde{L_{q1}}\right)_\alpha^{LB_2} = w, x, y, z \in R^{*Min} \frac{w\left(\frac{x}{y} / y + \frac{x}{z} / z\right)}{\left(1-\frac{w}{y}\right)}, \tag{23}$$

s.t. $x_\alpha^L \le x \le x_\alpha^U$, $w \in \lambda_1(\alpha)$, $y \in \mu_1(\alpha)$, $z \in \mu_2(\alpha)$

and

$$\left(\widetilde{L_{q1}}\right)_{\alpha}^{UB_2} = w, x, y, z \in R^{*Max} \frac{w\left(\frac{x}{y}/y+\frac{x}{z}/z\right)}{\left(1-\frac{w}{y}\right)}, \tag{24}$$

s.t. $x_{\alpha}^{L} \le x \le x_{\alpha}^{U}$, $w \in \lambda_1(\alpha)$, $y \in \mu_1(\alpha)$, $z \in \mu_2(\alpha)$

$$\left(\widetilde{L_{q1}}\right)_{\alpha}^{LB_3} = w, x, y, z \in R^{*Min} \frac{w\left(\frac{x}{y}/y+\frac{x}{z}/z\right)}{\left(1-\frac{w}{y}\right)}, \tag{25}$$

s.t. $y_{\alpha}^{L} \le y \le y_{\alpha}^{U}$, $w \in \lambda_1(\alpha)$, $x \in \lambda_2(\alpha)$, $z \in \mu_2(\alpha)$

and

$$\left(\widetilde{L_{q1}}\right)_{\alpha}^{UB_3} = w, x, y, z \in R^{*Max} \frac{w\left(\frac{x}{y}/y+\frac{x}{z}/z\right)}{\left(1-\frac{w}{y}\right)}, \tag{26}$$

s.t. $y_{\alpha}^{L} \le y \le y_{\alpha}^{U}$, $w \in \lambda_1(\alpha)$, $x \in \lambda_2(\alpha)$, $z \in \mu_2(\alpha)$

$$\left(\widetilde{L_{q1}}\right)_{\alpha}^{LB_4} = w, x, y, z \in R^{*Min} \frac{w\left(\frac{x}{y}/y+\frac{x}{z}/z\right)}{\left(1-\frac{w}{y}\right)}, \tag{27}$$

s.t. $z_{\alpha}^{L} \le z \le z_{\alpha}^{U}$, $w \in \lambda_1(\alpha)$, $x \in \lambda_2(\alpha)$, $y \in \mu_1(\alpha)$

and

$$\left(\widetilde{L_{q1}}\right)_{\alpha}^{UB_4} = w, x, y, z \in R^{*Max} \frac{w\left(\frac{x}{y}/y+\frac{x}{z}/z\right)}{\left(1-\frac{w}{y}\right)}, \tag{28}$$

s.t. $z_{\alpha}^{L} \le z \le z_{\alpha}^{U}$, $w \in \lambda_1(\alpha)$, $x \in \lambda_2(\alpha)$, $y \in \mu_1(\alpha)$

According to the definitions of $\lambda_1(\alpha)$, $\lambda_2(\alpha)$, $\mu_1(\alpha)$ and $\mu_2(\alpha)$ in equations (15)–(18), we can replace $w \in \lambda_1(\alpha)$, $x \in \lambda_2(\alpha)$, $y \in \mu_1(\alpha)$ and $z \in \mu_2(\alpha)$ with $w \in [w_{\alpha}^{L}, w_{\alpha}^{U}]$, $[w_{\alpha_1}^{L}, w_{\alpha_1}^{U}] \subseteq [w_{\alpha_2}^{L}, w_{\alpha_2}^{U}]$ and $z \in [z_{\alpha}^{L}, z_{\alpha}^{U}]$, respectively. We follow the suggestion of Zimmermann (2011) and Kaufmann (1975) to present the α-cut of *w*, *x*, *y* and *z* in nested form. The two possibility levels α_1 and α_2 imply $[w_{\alpha_1}^{L}, w_{\alpha_1}^{U}] \subseteq [w_{\alpha_2}^{L}, w_{\alpha_2}^{U}]$,

$[x^L_{\alpha_1}, x^U_{\alpha_1}] \subseteq [x^L_{\alpha_2}, x^U_{\alpha_2}]$, $[y^L_{\alpha_1}, y^U_{\alpha_1}] \subseteq [y^L_{\alpha_2}, y^U_{\alpha_2}]$, and $[z^L_{\alpha_1}, z^U_{\alpha_1}] \subseteq [z^L_{\alpha_2}, z^U_{\alpha_2}]$, when $0 < \alpha_1 < \alpha_2 \leq 1$.

The smallest value of $(\widetilde{L_{q1}})^{LB_1}_\alpha, (\widetilde{L_{q1}})^{LB_2}_\alpha, (\widetilde{L_{q1}})^{LB_3}_\alpha$ and $(\widetilde{L_{q1}})^{LB_4}_\alpha$ in equations (21), (23), (25) and (27) is the same. Conversely, the largest value of $(\widetilde{L_{q1}})^{UB_1}_\alpha, (\widetilde{L_{q1}})^{UB_2}_\alpha, (\widetilde{L_{q1}})^{UB_3}_\alpha$ and $(\widetilde{L_{q1}})^{UB_4}_\alpha$ in equations (22), (24), (26) and (28) is also the same way. In order to construct the membership function $\mu_{\widetilde{L_{q1}}}(z)$, it is necessary to determine the lower bound $(\widetilde{L_{q1}})^{LB}_\alpha$ and the upper bound $(\widetilde{L_{q1}})^{UB}_\alpha$ as:

$$\left(\widetilde{L_{q1}}\right)^{LB}_\alpha = w, x, y, z \in R^{*Min} \frac{w\left(\frac{x}{y}/y + \frac{x}{z}/z\right)}{\left(1-\frac{w}{y}\right)}, \tag{29}$$

s.t. $w^L_\alpha \leq w \leq w^U_\alpha$, $x^L_\alpha \leq x \leq x^U_\alpha$, $y^L_\alpha \leq y \leq y^U_\alpha$, and $z^L_\alpha \leq z \leq z^U_\alpha$,

and

$$\left(\widetilde{L_{q1}}\right)^{UB}_\alpha = w, x, y, z \in R^{*Max} \frac{w\left(\frac{x}{y}/y + \frac{x}{z}/z\right)}{\left(1-\frac{w}{y}\right)}, \tag{30}$$

s.t. $w^L_\alpha \leq w \leq w^U_\alpha$, $x^L_\alpha \leq x \leq x^U_\alpha$, $y^L_\alpha \leq y \leq y^U_\alpha$, and $z^L_\alpha \leq z \leq z^U_\alpha$.

At least one of *w*, *x*, *y* and *z* must be on the boundary of the constraints shown in equations (29) and (30). This model includes a set of mathematical programs and boundary constraints, which can be regarded as a special case of parametric NLP (Gal, 1979).

As α ranges over the interval (0, 1], it is of interest to examine how the optimal solutions change with different values for $w^L_\alpha, w^U_\alpha, x^L_\alpha, x^U_\alpha, y^L_\alpha, y^U_\alpha, z^L_\alpha$ and z^U_α, interval $[(\widetilde{L_{q1}})^{LB}_\alpha, (\widetilde{L_{q1}})^{UB}_\alpha]$ is a crisp closed interval which represents the α-cut of $\widetilde{L_{q1}}$. Again based on the extension principle and the convexity of a fuzzy number, we obtain $(\widetilde{L_{q1}})^{LB}_{\alpha_1} \geq (\widetilde{L_{q1}})^{UB}_{\alpha_2}$ and $(\widetilde{L_{q1}})^{LB}_{\alpha_1} \leq (\widetilde{L_{q1}})^{UB}_{\alpha_2}$ for $0 < \alpha_2 < \alpha_1 \leq 1$. In other words, an increase is observed in $(\widetilde{L_{q1}})^{LB}_\alpha$ increasing or $(\widetilde{L_{q1}})^{UB}_\alpha$ decreasing following an increase in α. Thus, these bounds can be used to develop the membership function $\mu_{\widetilde{L_{q1}}}(z)$.

Let us define an increasing function $(L_{q1})^{LB} : \alpha \to (L_{q1})^{LB}_\alpha$. If both $(L_{q1})^{LB}$ and $(L_{q1})^{UB}$ are invertible with respect to α, the membership function $\mu_{\widetilde{L_{q1}}}(z)$ can be represented by:

$$\mu_{\widetilde{L_{q1}}}(z) = \begin{cases} L(z), & \left(L_{q1}\right)^{LB}_{\alpha=0} \leq z \leq \left(L_{q1}\right)^{LB}_{\alpha=1}, \\ 1 & \left(L_{q1}\right)^{LB}_{\alpha=1} \leq z \leq \left(L_{q1}\right)^{UB}_{\alpha=1}, \\ R(z), & \left(L_{q1}\right)^{UB}_{\alpha=1} \leq z \leq \left(L_{q1}\right)^{UB}_{\alpha=0}. \end{cases} \tag{31}$$

where $L(z) = [(L_{q1})_\alpha^{LB}]^{-1}$ represent the left shape function and $R(z) = [(L_{q1})_\alpha^{UB}]^{-1}$ represent the right shape function, respectively. Nevertheless, it should be noted that under most cases $[(L_{q1})_\alpha^{LB}]^{-1}$ and $[(L_{q1})_\alpha^{UB}]^{-1}$ cannot be solved analytically. As a result, obtaining the shape of $\mu_{\widetilde{L_{q1}}}$ in closed form is difficult. To this end, we apply the set of intervals $\{[(L_{q1})_\alpha^{LB}]^{-1}, [(L_{q1})_\alpha^{UB}]^{-1} \mid 0 < \alpha \le 1\}$ to approximate the shape of $\mu_{\widetilde{L_{q1}}}(z)$ numerically. In the following section, we provide a solution procedure to compute the membership function for α, possibility levels. Similar approach is followed by obtaining the expression and corresponding values of membership function $\mu_{\widetilde{L_{q2}}}(z)$, which evaluates the number of customers in the second class. Although one crisp value is preferred for one of the system characteristics and this is obtained by Yager's (1981) ranking method since the Yager's index possesses the property of area compensation. We adopt this method for transforming the fuzzy values of system characteristics into crisp one to provide suitable values for system characteristics and the recommended suitable values of system characteristics are calculated by;

$$R(\tilde{a}) = \int_0^1 \frac{\left(a_\alpha^L + a_\alpha^U\right)}{2} d_\alpha \tag{32}$$

where $(\tilde{a})$ is a convex fuzzy number and (a_α^L, a_α^U) is the α-cut of $\tilde{a}$. This method is called robust ranking method that possesses the properties of compensation, linearity and additivity (Fortemps and Roubens, 1996).

4 Numerical example

To illustrate the practical application of this new approach to a priority single queuing model, we consider a single operator machine working on the production line with two priorities and different service rates. The arrival and service rates are represented as trapezoidal fuzzy numbers as: $\widetilde{\lambda_1} = [1, 2, 3, 4]$, $\widetilde{\lambda_2} = [5, 6, 7, 8]$, $\widetilde{\mu_1} = [10, 11, 12, 13]$ and $\widetilde{\mu_2} = [14, 15, 16, 17]$, respectively. The α-cut of the membership functions $\mu_{\widetilde{\lambda_1}}(w)$, $\mu_{\widetilde{\lambda_2}}(x)$, $\mu_{\widetilde{\mu_1}}(y)$ and $\mu_{\widetilde{\mu_2}}(z)$ are can be easily obtained to be $[1 + \alpha, 4 - \alpha]$, $[5 + \alpha, 8 - \alpha]$, $[10 + \alpha, 13 - \alpha]$ and $[14 + \alpha, 17 - \alpha]$ respectively.

We further construct the membership function of upper bound and lower bound of the α-cut of $\widetilde{L_{q1}}$ and $\widetilde{L_{q2}}$. When w and x takes its lower bound also y and z take their upper bound, the membership function through the equation with constraints given with equations (29) yields the following results.

$$\left(L_{q1}\right)_\alpha^{LB} = \frac{\alpha^4 - 26\alpha^3 + 120\alpha^2 + 714\alpha + 567}{\alpha^4 - 53\alpha^3 + 1,013\alpha^2 + 8,143\alpha + 22,542}, \tag{33}$$

While, when w and x takes it's the upper bound also, y and z take their lower bound the membership function through the equation with constriants given with equation (30) this yields the equation as:

$$\left(L_{q1}\right)_{\alpha}^{UB} = \frac{\alpha^4 + 14\alpha^3 - 60\alpha^2 - 840\alpha + 3{,}168}{\alpha^4 + 41\alpha^3 + 590\alpha^2 + 3{,}388\alpha + 5{,}880}, \tag{34}$$

By the same argument and equation (20), the membership function of $\widetilde{L_{q2}}$ is given as:

$$\left(L_{q2}\right)_{\alpha}^{LB} = \frac{\alpha^4 - 22\alpha^3 + 12\alpha^2 + 1{,}302\alpha + 2{,}835}{\alpha^4 - 61\alpha^3 + 1{,}245\alpha^2 + 10{,}063\alpha + 27{,}438}, \tag{35}$$

$$\left(L_{q2}\right)_{\alpha}^{UB} = \frac{\alpha^4 + 10\alpha^3 - 132\alpha^2 - 888\alpha + 6{,}336}{\alpha^4 + 49\alpha^3 + 750\alpha^2 + 4{,}132\alpha + 6{,}888}. \tag{36}$$

The inverse functions of $(\widetilde{L_{q1}})_{\alpha}^{LB}$, $(\widetilde{L_{q1}})_{\alpha}^{UB}$, $(\widetilde{L_{q2}})_{\alpha}^{LB}$ and $(\widetilde{L_{q2}})_{\alpha}^{UB}$ exists which is given as:

$$\mu_{\widetilde{L_{q1}}}(z) = \begin{cases} L(z), & \frac{189}{7{,}514} \le z \le \frac{688}{15{,}823} \\ 1, & \frac{688}{15{,}823} \le z \le \frac{761}{3{,}300} \\ R(z), & \frac{761}{3{,}300} \le z \le \frac{132}{245} \end{cases} \tag{37}$$

and

$$\mu_{\widetilde{L_{q2}}}(z) = \begin{cases} L(z), & \frac{945}{9{,}146} \le z \le \frac{2{,}064}{19{,}343} \\ 1, & \frac{2{,}064}{19{,}343} \le z \le \frac{2{,}449}{3{,}940} \\ R(z), & \frac{2{,}449}{3{,}940} \le z \le \frac{264}{287} \end{cases} \tag{38}$$

See Table 1 for the α-cut interval queue length both two classes. Likewise, the numerical results of the membership function given at different values for α, are shown in Figures 1 and 2.

Table 1 α-cut interval queue length both two classes

α	0.0	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
$(L_{q1})_{\alpha}^{LB}$	0.0025	0.0027	0.0029	0.0031	0.0033	0.0035	0.0037	0.0038	0.0040	0.0042	0.0043
$(L_{q1})_{\alpha}^{UB}$	0.5387	0.4953	0.4554	0.4188	0.3850	0.3539	0.3252	0.2987	0.2742	0.2515	0.2306
$(L_{q2})_{\alpha}^{LB}$	0.1033	0.1042	0.1049	0.1055	0.1060	0.1062	0.1063	0.1064	0.1065	0.1066	0.1067
$(L_{q2})_{\alpha}^{UB}$	0.9198	0.8792	0.8417	0.8069	0.7746	0.7446	0.7166	0.6905	0.6660	0.6431	0.6215

Figure 1 The membership function of the expected numbers of customers in the queue for Class 1

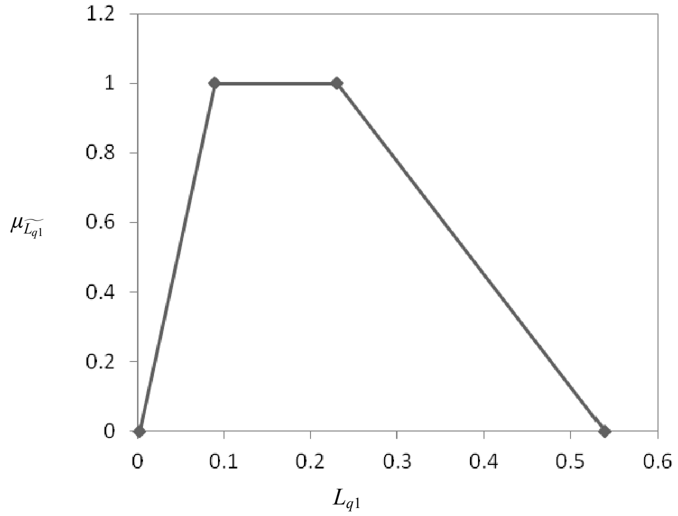
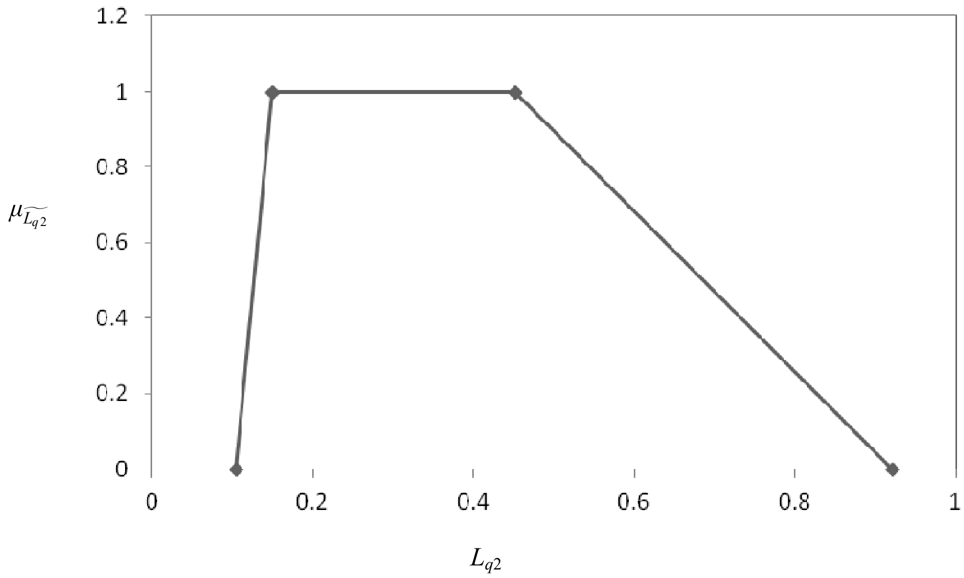


Figure 2 The membership function of the expected numbers of customers in the queue for Class 2



However, we wish to obtain one crisp value inside the closed intervals. Therefore, we apply Yager's ranking index to defuzzify the fuzzy values and then transforming these fuzzy values into crisp values. To obtain $\widetilde{L_{q1}}$ and $\widetilde{L_{q2}}$, we refer to equations (33) and (34) for the lower bound and upper bound of the first priority class, and in the same vein, refer to equations (35) and (36) to obtain the bounds for the second priority class. By using equation (32), we obtain:

$$R\left(\widetilde{L_{q1}}\right)=\int_0^1 \frac{1}{2}\left[\frac{\alpha^4-26\alpha^3+120\alpha^2+714\alpha+567}{\alpha^4-53\alpha^3+1,013\alpha^2+8,143\alpha+22,542}\right.$$

$$\left.+\frac{\alpha^4+14\alpha^3-60\alpha^2-840\alpha+3,168}{\alpha^4+41\alpha^3+590\alpha^2+3,388\alpha+5,880}\right]d_\alpha \tag{39}$$

$$\therefore R\left(\widetilde{L_{q1}}\right)=0.20853$$

and

$$R\left(\widetilde{L_{q2}}\right)=\int_0^1 \frac{1}{2}\left[\frac{\alpha^4-22\alpha^3+12\alpha^2+1,302\alpha+2,835}{\alpha^4-61\alpha^3+1,245\alpha^2+10,063\alpha+27,438}\right.$$

$$\left.+\frac{\alpha^4+10\alpha^3-132\alpha^2-888\alpha+6,336}{\alpha^4+49\alpha^3+750\alpha^2+4,132\alpha+6,888}\right]d_\alpha \tag{40}$$

$$\therefore R\left(\widetilde{L_{q2}}\right)=0.40599$$

From observation, it can be seen that the exact real value of L_{q1} is 0.2085 in priority class one is less than the result in priority class two represented by L_{q2} is 0.4059. This implies that L_q is 0.6144, this is the value for the total queue length of customers in the queue with the two class of high and low priority. This leads to obtain new information to the service system that the real queue length of customer in class one inside the system reflects double queue length of customer in Class 2. The equations (39) and (40) are important for the derivation of the inverse functions and the equations also make it easy to deal with evaluating the system within the closed form interval [0, 1]. Table 1 presented the α-cut range of the possibility queue length for each class of priority in the single queueing model.

5 Results and discussion

The following observations can be drawn from Table 1 for the α-cut queue length of both priority classes:

- At $\alpha = 0$, the range of the fuzzy queue length of L_{q1} (priority class one) is [0.0025, 0.0043] and [0.1033, 0.1067] for L_{q2} (priority class two). This implies that the number of customers in the queue with different service rates cannot exceed 0.5387 and 0.9198 or fall below 0.0025 and 0.1033.
- At $\alpha = 1$, the range of the queue length of priority class one is [0.0895, 0.2306] and [0.9198, 0.6215] for the priority class two. This implies that the number of customers for both classes fall between 0.0043 and 0.2306, and, 0.1067 and 0.6215 respectively.

This numerical example, has given useful insights for a queueing system having unequal service rates. This includes the information presented for obtaining the expected queue length of customers for each class with single channel model.

6 Conclusions

A range of real life situations can be categorised as fuzzy queues with unequal service rates, such as in the production line of manufacturing system. Note that the main aim of any system is to obtain optimal system performance measures and this paper leads to adequately present an approach for obtaining the range of the expected number of customers in the queue which will assist managers to properly plan the affairs of the production. Also, this paper has presented a unique approach for the development of the membership function of fuzzy queues, where the unequal service rates are categorised using priority class and represented as fuzzy numbers. The methodology adopted α-cut interval to obtain crisp values inside the closed interval, also the numerical example assumed the arrival and service rates to be trapezoidal fuzzy numbers. For future work it can be highlighted to extend these single fuzzy queues into multiple fuzzy queues whether having equal or different service rates within the queueing system.

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