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A hybrid technique for solving fractional delay variational problems by the shifted Legendre polynomials

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ABSTRACT

This study presents a practical method for solving fractional order delay variational problems. The fractional derivative is given in the Caputo sense. The suggested approach is based on the Laplace transform and the shifted Legendre polynomials by approximating the candidate function by the shifted Legendre series with unknown coefficients yet to be determined. The proposed method converts the fractional order delay variational problem into a set of (n + 1) algebraic equations, where the solution to the resultant equation provides us the unknown coefficients of the terminated series that have been utilized to approximate the solution to the considered variational problem. Illustrative examples are given to show that the recommended approach is applicable and accurate for solving such kinds of problems.

1. Introduction

Fractional calculus, also known as calculus of fractional order or fractional differential calculus, is a section of mathematics that transacts with differentiation and integration of fractional order. It generalizes the concepts of differentiation and integration to include fractional powers. Since fractional calculus is an active area of research the mathematicians and scientists continue to explore its theoretical foundations and practical applications. It provides a powerful tool for modeling and analyzing systems with memory effects, long-range interactions, and fractal-like behavior in Engineering,[1,2] Biology,[3] Epidemiology,[4] Chemistry,[5] Physics,[6] Control Theory,[7–9] electrical and electromechanical systems,[10,11] Fluid Mechanics,[12] viscoplasticity,[13] etc.[14–19]

The distinctiveness of fractional calculus, which may be seen as the reason for its effectiveness in applications for problems in the real world, is that the abundance of fractional operators enables researchers to select the best appropriate one to simulate the problem under research. There has been an upsurge in recent years in research on these fractional computation issues. The research into these concerns are mostly centered on the use of fractional derivatives, particularly Riemann–Liouville(R–L) and Caputo fractional derivatives, in place of classical derivatives.

It is widely known that time delay act an important function in designing several processes and dynamical systems in real-world problems.[20,21] Time-delayed equations frequently occur in different branches of research and several kinds of engineering especially population, electrical engineering, mechanical engineering, manufacturing process and nervous networks.[22–24]

The delay fractional equation is used to represent the mathematical model for cancer development, See Refs. 25, 26. It might be maintained that variation problems with delayed arguments occupy an important space. The fractional calculus of variations include time delay is still the subject of very few studies in the literature.

Variational methods such as Ritz technique[27] and variational iteration method[28–31] have been, and continue to be, popular tools for nonlinear analysis. For example, the soliton solutions and their evolution of lots of nonlinear equations were accurately captured by the variational approximation method, which always substitutes some ansatzs into the obtained Lagrange functional, and find the variational parameters by solving the corresponding Euler–Lagrange equations.[27–31] When contrasted with other analytical or numerical methods, variational-based methods show a lot of advantages.[30,31]

Firstly, they can be used to investigate practical problems from a global perspective, and provide physical insight into the nature of the solutions. Secondly, they suggest an energy conservation for the whole solution domain, and require much less strong local differentiability of each variable than the methods, which solve PDEs directly. Last but not the least, the obtained solutions are the best among all the possible

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trial-functions. Because variational principles are the theoretical basis for many kinds of variational-based methods, it is very important to seek explicit variational formulations for nonlinear and complex PDEs, which is a nontrivial problem.

Recently, many scientists have made many attempts and great success for constructing different kinds of variational principles in various fields such as fluid dynamics, meteorology, ocean, mathematical biology, solid state physics, and plasma physics.[32–37]

The fractal variational principle in a fractal space becomes a hot topic in mathematics, mechanics and physics, it plays a significant role in variational-based numerical methods and analytical methods.[38–43] Additionally it is the most effective tool to the establishment of a fractal differential equation for a porous medium, or an unsmooth boundary problem, or a discontinuous problem[44]

In this paper, we reduce the time-delay problem to a nondelay system of algebraic equations using the properties of the Legendre polynomials (LPs) and Laplace transformations.

Numerous scientific and technical fields have found extensive use for approximation by orthogonal families of functions. Orthogonal functions that are most often used are Legendre polynomials,[45,46] Legendre wavelet,[47] Chebyshev polynomials,[48] Chebyshev wavelet,[49] Euler polynomials,[50] Laguerre series,[51] fractional-order Legendre wavelets.[52]

This work investigates the solution of the fractional order time-delayed variational problems by approximating the functional in terms of shifted Legendre polynomials with the aid of the Laplace transform because of computational efficiency of the Legendre polynomials also the combination of Legendre polynomials and the Laplace transform are more accurate than other kinds of polynomials and finally the minimization procedure has been done.

2. Preliminaries

Definition 2.1. The (R–L) fractional integral operator I_τ^η of order $\eta > 0$ of a function $\mathscr{Y} \in C_\eta, \eta \geq -1$ can be expressed as follows[53]:

$$I_\tau^\eta \mathscr{Y}(\tau) = \begin{cases} \dfrac{1}{\Gamma(\eta)} \displaystyle\int_0^\tau (\tau - v)^{\eta-1} \mathscr{Y}(v) dv, & \eta > 0, \ v > 0, \\ \mathscr{Y}(\tau), & \eta = 0. \end{cases} \tag{2.1}$$

Definition 2.2. The Caputo fractional derivative ${}_0^cD_\tau^\eta$ of order $\eta > 0$ of a function $\mathscr{Y} \in C_\eta^m, \eta \geq -1$ is defined as follows[53]:

$$ {}_0^cD_\tau^\eta \mathscr{Y}(\tau) = \begin{cases} \dfrac{1}{\Gamma(m-\eta)} \displaystyle\int_0^\tau \dfrac{\mathscr{Y}^{(m)}(v)}{(\tau - v)^{\eta+1-m}} dv, & m-1 < \eta < m, \ v > 0, \ m \in N, \\ \dfrac{d^m}{d\tau^m} \mathscr{Y}(\tau), & \eta = m, \end{cases} \tag{2.2}$$

Below, we list some relations between the Caputo's and (R–L) operators as:

(1) $I_\tau^\eta \tau^\mu = \left(\frac{\Gamma(\mu+1)}{\Gamma(\mu+\eta+1)}\right) \tau^{\eta+\mu}, \qquad \mu > -1, \quad \eta > 0.$

(2) ${}_0^cD_\tau^\eta \tau^\mu = \left(\frac{\Gamma(\mu+1)}{\Gamma(\mu-\eta+1)}\right) \tau^{\mu-\eta}, \qquad \mu > -1, \quad \eta > 0.$

(3) ${}_0^cD_\tau^\eta (I_\tau^\eta \mathscr{Y}(\tau)) = \mathscr{Y}(\tau).$

(4) $I_\tau^\eta ({}_0^cD_\tau^\eta \mathscr{Y}(\tau)) = \mathscr{Y}(\tau) - \sum_{\kappa=0}^{n-1} \frac{\mathscr{Y}^{(\kappa)}(0)}{\kappa!} \tau^\kappa, \qquad \tau \geq 0, \quad n-1 < \eta < n.$

(5) $\mathscr{L}\left\{{}_0^cD_\tau^\eta \mathscr{Y}(\tau)\right\} = s^\eta \mathcal{L}\left\{\mathscr{Y}(\tau)\right\} - \sum_{\kappa=0}^{n-1} s^{\eta-\kappa-1} \mathscr{Y}^{(\kappa)}(0)$, where $\mathscr{L}$ is the Laplace transform operator.

Definition 2.3. The Atangana–Baleanu fractional derivatives in the caputo sense(ABC derivative) of order η for given function $\mathscr{Y}(\tau) \in H^1(a,b), b > a$, $\eta > 0$ (where A denotes Atangana, B denotes Baleanu and C denotes Caputo type) with base point a is defined at a point $\tau \in (a,b)$.

1. The left Atangana–Baleanu–Caputo derivative:

$${}_a^{ABC}D_\tau^\eta \ \mathscr{Y}(\tau) = \frac{B(\eta)}{1-\eta} \int_a^\tau \acute{\mathscr{Y}}(\tau) \ E_\eta[-\gamma \frac{(\tau - s)^\eta}{1-\eta}] \, ds, \tag{2.3}$$

where $\gamma = \frac{\eta}{(1-\eta)}$, and $B(\eta)$ being a normalization function satisfying

$$B(\eta) = (1-\eta) + \frac{\eta}{\Gamma(\eta)}, \tag{2.4}$$

where $B(0) = B(1) = 1$, $E_\eta(.)$ stands for the Mittag-Leffler function defined by

$$E_{\eta,\beta}(z) = \sum_{\kappa=0}^{\infty} \frac{z^\kappa}{\Gamma(\kappa a + \beta)}, \quad E_{\eta,1}(z) = E_\eta(z), \quad z \in C, \tag{2.5}$$

which is an entire function on the complex plane ans $\Gamma(.)$ denotes the Euler's gamma function defined as

$$\Gamma(z) = \int_0^\infty \tau^{z-1} e^{-\tau} d\tau, \quad R(z) > 0. \tag{2.6}$$

2. The right Atangana–Baleanu–Caputo derivative:

$${}_b^{ABC}D_\tau^\eta \ \mathscr{Y}(\tau) = \frac{B(\eta)}{1-\eta} \int_\tau^b \acute{\mathscr{Y}}(s) \ E_\eta[-\eta \frac{(s-\tau)^\eta}{1-\eta}] \, ds. \tag{2.7}$$

3. Shifted Legendre polynomials (ShLPs)

Suppose that the (LPs) of degree i are indicated by $\mathcal{L}_i(t)$ that defined on the interval (−1,1), by the recurrence formulae[54]

$$\mathcal{L}_{i+1}(t) = \frac{2i+1}{i+1} t \mathcal{L}_i(t) - \frac{i}{i+1} \mathcal{L}_{i-1}(t), \qquad i = 1, 2, \dots,$$

$$\mathcal{L}_0(t) = 1, \qquad \mathcal{L}_1(t) = t,$$

then by $t = 2\tau - 1$, (LPs) that defined on the interval (0,1) will be called (ShLPs) and is denoted by $\mathcal{L}_i^*(\tau)$ and it is generated using the following recurrence formulae

$$\mathcal{L}_{i+1}^*(\tau) = \frac{2i+1}{i+1} (2\tau - 1) \mathcal{L}_i^*(\tau) - \frac{i}{i+1} \mathcal{L}_{i-1}^*(\tau), \qquad i = 1, 2, \dots,$$

$$\mathcal{L}_0^*(\tau) = 1, \qquad \mathcal{L}_1^*(\tau) = 2\tau - 1.$$

The orthogonality relation is defined by

$$\int_0^1 \mathcal{L}_j^*(\tau) \mathcal{L}_i^*(\tau) d\tau = \begin{cases} \frac{1}{2i+1}, & for \ \ i = j, \\ 0, & for \ \ i \neq j. \end{cases}$$

The (ShLPs) $\mathcal{L}_i^*(\tau)$ of degree i can be expressed explicitly analytically as

$$\mathcal{L}_i^*(\tau) = \sum_{\kappa=0}^{i} (-1)^{i+\kappa} \frac{(i+\kappa)! \tau^\kappa}{(i-\kappa)! (\kappa!)^2},$$

where

$$\mathcal{L}_i^*(0) = (-1)^i, \qquad \mathcal{L}_i^*(1) = 1.$$

(ShLPs) can be used to represent any function $f(\tau) \in L_2[0,1]$ as

$$f(\tau) = \sum_{i=0}^{\infty} c_i \mathcal{L}_i^*(\tau),$$

where c_i given by

$$c_i = (2i+1) \int_0^1 f(\tau) \mathcal{L}_i^*(\tau) d\tau.$$

If we use the first $(n+1)$-terms to approximate $f(\tau)$, we may write

$$f_n(\tau) = \sum_{i=0}^{n} c_i \mathcal{L}_i^*(\tau).$$

4. The direct approach

In this section, we consider the fractional order delay variational problems (FODVPs)

$$\mathcal{J}[\mathscr{Y}]=\int_a^b \mathcal{F}(\tau,\mathscr{Y}(\tau),\mathscr{Y}(\tau-\nu),{}_0^cD_\tau^\eta\mathscr{Y}(\tau),I_\tau^\eta\mathscr{Y}(\tau))d\tau, \tag{4.1}$$

subject to

$$\mathscr{Y}(b)=\mathscr{Y}_b \in R,$$

and

$$\mathscr{Y}(\tau)=\varphi(\tau) \quad for\ any \ \ \tau\in[a-\nu,a],$$

where φ is a given function.

Based on the (ShLPs), this approach approximates the fractional order derivative ${}_0^cD_\tau^\eta\mathscr{Y}(\tau)$ as follows:

$${}_0^cD_\tau^\eta\mathscr{Y}(\tau)=\sum_{i=0}^{n}c_i\mathcal{L}_i^*(\tau), \tag{4.2}$$

where $\mathcal{L}_i^*(\tau)$ are (ShLPs) and the unknown coefficients to be determined are $c_i, i=0,1,\dots,n$. When both sides of Eq. (4.2) are subjected to the Laplace transform, we find

$$\mathscr{L}\left\{{}_0^cD_\tau^\eta\mathscr{Y}(\tau)\right\}=\mathscr{L}\left\{\sum_{i=0}^{n}c_i\mathcal{L}_i^*(\tau)\right\}. \tag{4.3}$$

The Laplace transform of the fractional order derivative (see property (5)), yields

$$s^\eta\mathscr{L}\{\mathscr{Y}(\tau)\}-\sum_{\kappa=0}^{n-1}s^{\eta-\kappa-1}\mathscr{Y}^{(\kappa)}(0)=\mathscr{L}\left\{\sum_{i=0}^{n}c_i\mathcal{L}_i^*(\tau)\right\}, \tag{4.4}$$

or

$$\mathscr{L}\{\mathscr{Y}(\tau)\}=\frac{1}{s^\eta}\sum_{\kappa=0}^{n-1}s^{\eta-\kappa-1}\mathscr{Y}^{(\kappa)}(0)+\frac{1}{s^\eta}\mathscr{L}\left\{\sum_{i=0}^{n}c_i\mathcal{L}_i^*(\tau)\right\}. \tag{4.5}$$

Applying inverse Laplace transform on both sides of Eq. (4.5), gives

$$\mathscr{Y}(\tau)=\mathscr{L}^{-1}\left\{\frac{1}{s^\eta}\sum_{\kappa=0}^{n-1}s^{\eta-\kappa-1}\mathscr{Y}^{(\kappa)}(0)+\frac{1}{s^\eta}\mathscr{L}\left\{\sum_{i=0}^{n}c_i\mathcal{L}_i^*(\tau)\right\}\right\}. \tag{4.6}$$

If Eqs. (4.2) and (4.6) are substituted into Eq. (4.1), giving us

$$\begin{aligned}\mathcal{J}[\mathscr{Y}]=\int_a^b\mathcal{F}\Bigg[&\tau,\mathscr{L}^{-1}\left\{\frac{1}{s^\eta}\sum_{\kappa=0}^{n-1}s^{\eta-\kappa-1}\mathscr{Y}^{(\kappa)}(0)\right.\\ &\left.+\frac{1}{s^\eta}\mathscr{L}\left\{\sum_{i=0}^{n}c_i\mathcal{L}_i^*(\tau)\right\}\right\},\varphi(\tau-\nu) \\ &,\sum_{i=0}^{n}c_i\mathcal{L}_i^*(\tau),\frac{1}{\Gamma(\eta)}\int_0^\tau(\tau-\nu)^{\eta-1}\Bigg(\mathscr{L}^{-1}\left\{\frac{1}{s^\eta}\sum_{\kappa=0}^{n-1}s^{\eta-\kappa-1}\mathscr{Y}^{(\kappa)}(0)\right.\\ &\left.+\frac{1}{s^\eta}\mathscr{L}\left\{\sum_{i=0}^{n}c_i\mathcal{L}_i^*(\nu)\right\}\right\}\Bigg)d\nu\Bigg]d\tau.\end{aligned} \tag{4.7}$$

We define a new functional $\mathcal{V}[\mathscr{Y}]$ by

$$\mathcal{V}[\mathscr{Y}]=\mathcal{J}[\mathscr{Y}]-\lambda\left(\mathscr{Y}(b)-\mathscr{Y}_b\right), \tag{4.8}$$

where λ is a Lagrange multiplier.

By using Eq. (4.6) into Eq. (4.8), yields

$$\begin{aligned}\mathcal{V}(c_0,c_1,c_2,\dots,c_n)=\int_a^b\mathcal{F}\Bigg[&\tau,\mathscr{L}^{-1}\left\{\frac{1}{s^\eta}\sum_{\kappa=0}^{n-1}s^{\eta-\kappa-1}\mathscr{Y}^{(\kappa)}(0)\right.\\ &\left.+\frac{1}{s^\eta}\mathscr{L}\left\{\sum_{i=0}^{n}c_i\mathcal{L}_i^*(\tau)\right\}\right\},\varphi(\tau-\nu)\\ &,\sum_{i=0}^{n}c_i\mathcal{L}_i^*(\tau),\frac{1}{\Gamma(\eta)}\int_0^\tau(\tau-\nu)^{\eta-1}\Bigg(\mathscr{L}^{-1}\left\{\frac{1}{s^\eta}\sum_{\kappa=0}^{n-1}s^{\eta-\kappa-1}\mathscr{Y}^{(\kappa)}(0)\right.\\ &\left.+\frac{1}{s^\eta}\mathscr{L}\left\{\sum_{i=0}^{n}c_i\mathcal{L}_i^*(\nu)\right\}\right\}\Bigg)d\nu\Bigg]d\tau\\ &-\lambda\left(\mathscr{L}^{-1}\left\{\frac{1}{s^\eta}\sum_{\kappa=0}^{n-1}s^{\eta-\kappa-1}\mathscr{Y}^{(\kappa)}(0)+\frac{1}{s^\eta}\mathscr{L}\left\{\sum_{i=0}^{n}c_i\mathcal{L}_i^*(b)\right\}\right\}-\mathscr{Y}_b\right).\end{aligned}$$

In Eq. (4.8), the initial extremization of the (FODVPs) is transformed into the extremization of a functional with a finite set of variables. When we set the partial derivatives of $\mathcal{V}$ with respect to c_i to zero, we obtain

$$\frac{\partial\mathcal{V}}{\partial c_i}=0, \qquad i=0,1,2,\dots,n.$$

One can arrive at the desired answer of Eq. (4.1) by solving for c_i and substituting into Eq. (4.6).

5. Numerical examples

We make the following examples into consideration to illustrate the effectiveness and precision of the suggested approach.

Example 5.1. Consider the (FODVPs) of the form

$$\mathcal{J}[\mathscr{Y}]=\int_0^2\left[\left({}_0^cD_\tau^\eta\mathscr{Y}(\tau)-\Gamma(\eta+2)\tau\right)^2+\gamma(\tau)+\left(\mathscr{Y}'(\tau-1)-\mathscr{Y}'_\eta(\tau-1)\right)^2\right]d\tau, \tag{5.1}$$

where

$$\gamma(\tau)=\int_0^\tau\left(\mathscr{Y}(t)-t^{\eta+1}\right)^2dt, \tag{5.2}$$

known on the $C^1[-1,2]$ set with the given constraints

$$\begin{cases}\mathscr{Y}(2)=2^{\eta+1},\\ \mathscr{Y}(\tau)=0, \quad for\ \ all \ \ \tau\in[-1,0],\end{cases} \tag{5.3}$$

where the closed form (see Ref. 55) is

$$\mathscr{Y}_\eta(\tau)=\begin{cases}0, & \tau\in[-1,0],\\ \tau^{\eta+1}, & \tau\in[0,2].\end{cases} \tag{5.4}$$

Consider, for $n=2$

$${}_0^cD_\tau^\eta\mathscr{Y}(\tau)=\sum_{i=0}^{2}c_i\mathcal{L}_i^*(\tau). \tag{5.5}$$

Therefore,

$$\begin{aligned}\mathscr{Y}(\tau)&=\mathscr{L}^{-1}\left\{\frac{1}{s^\eta}\sum_{\kappa=0}^{n-1}s^{\eta-\kappa-1}\mathscr{Y}^{(\kappa)}(0)+\frac{1}{s^\eta}\mathscr{L}\left\{\sum_{i=0}^{n}c_i\mathcal{L}_i^*(\tau)\right\}\right\}\\ &=\mathscr{L}^{-1}\left\{\frac{1}{s^\eta}\mathscr{L}\left\{c_0\mathcal{L}_0^*(\tau)+c_1\mathcal{L}_1^*(\tau)+c_2\mathcal{L}_2^*(\tau)\right\}\right\}\\ &=\mathscr{L}^{-1}\left\{\frac{c_0}{s^{\eta+1}}+\frac{2c_1}{s^{\eta+2}}-\frac{c_1}{s^{\eta+1}}+\frac{12c_2}{s^{\eta+3}}-\frac{6c_2}{s^{\eta+2}}+\frac{c_2}{s^{\eta+1}}\right\}.\end{aligned}$$

Then,

$$\begin{aligned}\mathscr{Y}(\tau)=&\frac{c_0}{\Gamma(\eta+1)}\tau^\eta+\frac{2c_1}{\Gamma(\eta+2)}\tau^{\eta+1}-\frac{c_1}{\Gamma(\eta+1)}\tau^\eta+\frac{12c_2}{\Gamma(\eta+3)}\tau^{\eta+2}\\ &-\frac{6c_2}{\Gamma(\eta+2)}\tau^{\eta+1}+\frac{c_2}{\Gamma(\eta+1)}\tau^\eta.\end{aligned} \tag{5.6}$$

Substituting Eqs. (5.5) and (5.6) into Eq. (5.1), we get

$$\begin{aligned}J[\mathscr{Y}(\tau)]=\int_0^2&\left[\left(\left(c_0\mathcal{L}_0^*(\tau)+c_1\mathcal{L}_1^*(\tau)+c_2\mathcal{L}_2^*(\tau)\right)-\Gamma(\eta+2)\tau\right)^2\right.\\ &+\left((\eta+1)(\tau-1)^\eta\right)^2+\end{aligned}$$

$$\int_0^\tau\Bigg(\Big(\frac{c_0}{\Gamma(\eta+1)}t^\eta+\frac{2c_1}{\Gamma(\eta+2)}t^{\eta+1}-\frac{c_1}{\Gamma(\eta+1)}t^\eta+\frac{12c_2}{\Gamma(\eta+3)}t^{\eta+2}$$

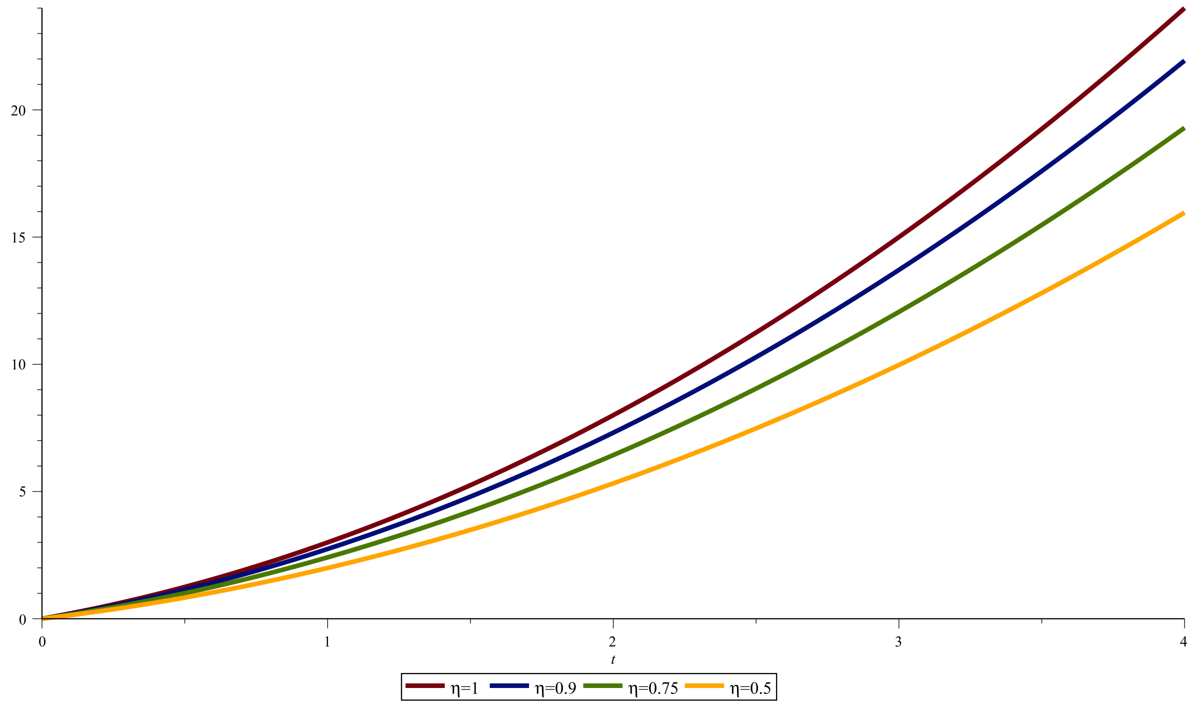

Fig. 1. The approximate solution of Example 5.1.

Table 1
(MAE) at various values of η.

η	(MAE)
0.9	3.109×10^{-15}
0.75	1.332×10^{-15}
0.5	1.332×10^{-15}

$$-\frac{6c_2}{\Gamma(\eta+2)}t^{\eta+1}\frac{c_2}{\Gamma(\eta+1)}t^{\eta}) - t^{\eta+1}\Big)^2 dx\Bigg] dt.$$

Now, we define

$$\mathcal{V}(c_0, c_1, c_2, \dots, c_n) = \mathcal{J}[\mathscr{Y}] - \lambda\left[\mathscr{Y}(2) - 2^{\eta+1}\right].$$

Differentiating $\mathcal{V}$ with regards to c_i and putting them equal to zero, then we get a system of $(n+1)$ algebraic equations with $(n+1)$ undetermined coefficients c_i.

Therefore the solution of the problem (5.1)–(5.4), now becomes ready by substituting the determined coefficients c_i, into Eq. (5.6).

Table 1 shows the maximum absolute errors(MAE) related to the problem (5.1)–(5.4) at different values of η.

Fig. 1 represent the approximate solution of problem (5.1)–(5.4) at various values of η.

Example 5.2.

$$\mathcal{J}[\mathscr{Y}] = \int_0^2 \left[\left({}_0^cD_\tau^\eta\mathscr{Y}(\tau) - f_1\right)^2 + \left(I_\tau^{\wp}\mathscr{Y}(\tau) - f_2\right)^2 + \gamma(\tau) + \left(\mathscr{Y}'(\tau-1) - \mathscr{Y}'_\eta(\tau-1)\right)^2\right] d\tau, \tag{5.7}$$

where

$$\gamma(\tau) = \int_0^\tau \left(\mathscr{Y}(t) - I_t^{\wp}\mathscr{Y}(t) + f_2 - 2t^{\frac{5}{2}}\right)^2 dt, \tag{5.8}$$

known on the $C^1[-1,2]$ set with the given constraints

$$\begin{cases} \mathscr{Y}(2) = 2^{\frac{7}{2}}, \\ \mathscr{Y}(\tau) = 0, \quad for\ all \quad \tau \in [-1,0], \end{cases} \tag{5.9}$$

and

$$f_1 = \frac{2\Gamma\left(\frac{7}{2}\right)\tau^{\frac{5}{2}-\eta}}{\Gamma\left(\frac{7}{2}-\eta\right)}, \qquad f_2 = \frac{2\Gamma\left(\frac{7}{2}\right)\tau^{\frac{5}{2}+\wp}}{\Gamma\left(\frac{7}{2}+\wp\right)}, \tag{5.10}$$

where the closed form is:

$$\mathscr{Y}_\eta(\tau) = \begin{cases} 0, & \tau \in [-1,0], \\ 2\tau^{\frac{5}{2}}, & \tau \in [0,2]. \end{cases} \tag{5.11}$$

Now, let

$${}_0^cD_\tau^\eta\mathscr{Y}(\tau) = \sum_{i=0}^{2} c_i\mathcal{L}_i^*(\tau), \tag{5.12}$$

and

$$I_\tau^{\wp}\mathscr{Y}(\tau) = \frac{c_2}{2}\tau^4 + \left(\frac{c_1}{3} - c_2\right)\tau^3 + \left(\frac{c_0 - c_1 + c_2}{2}\right)\tau^2. \tag{5.13}$$

Therefore,

$$\mathscr{Y}(\tau) = \mathscr{L}^{-1}\left\{\frac{c_0}{s^{\eta+1}} + \frac{2c_1}{s^{\eta+2}} - \frac{c_1}{s^{\eta+1}} + \frac{12c_2}{s^{\eta+3}} - \frac{6c_2}{s^{\eta+2}} + \frac{c_2}{s^{\eta+1}}\right\},$$

then

$$\mathscr{Y}(\tau) = \frac{c_0}{\Gamma(\eta+1)}\tau^\eta + \frac{2c_1}{\Gamma(\eta+2)}\tau^{\eta+1} - \frac{c_1}{\Gamma(\eta+1)}\tau^\eta + \frac{12c_2}{\Gamma(\eta+3)}\tau^{\eta+2} - \frac{6c_2}{\Gamma(\eta+2)}\tau^{\eta+1} + \frac{c_2}{\Gamma(\eta+1)}\tau^\eta. \tag{5.14}$$

Substituting Eqs. (5.12)–(5.14) into Eq. (5.7), we get

$$\mathcal{J}[\mathscr{Y}] = \int_0^2 \Bigg[\Bigg(\left(c_0\mathcal{L}_0^*(\tau) + c_1\mathcal{L}_1^*(\tau) + c_2\mathcal{L}_2^*(\tau)\right) - \frac{2\Gamma\left(\frac{7}{2}\right)\tau^{\frac{5}{2}-\eta}}{\Gamma\left(\frac{7}{2}-\eta\right)}\Bigg)^2 + \Bigg(\left(\frac{c_2}{2}\tau^4 + \left(\frac{c_1}{3} - c_2\right)\tau^3 + \left(\frac{c_0 - c_1 + c_2}{2}\right)\tau^2\right) - \frac{2\Gamma\left(\frac{7}{2}\right)\tau^{\frac{5}{2}+\wp}}{\Gamma\left(\frac{7}{2}+\wp\right)}\Bigg)^2 + \left((\eta+1)(\tau-1)^\eta\right)^2$$

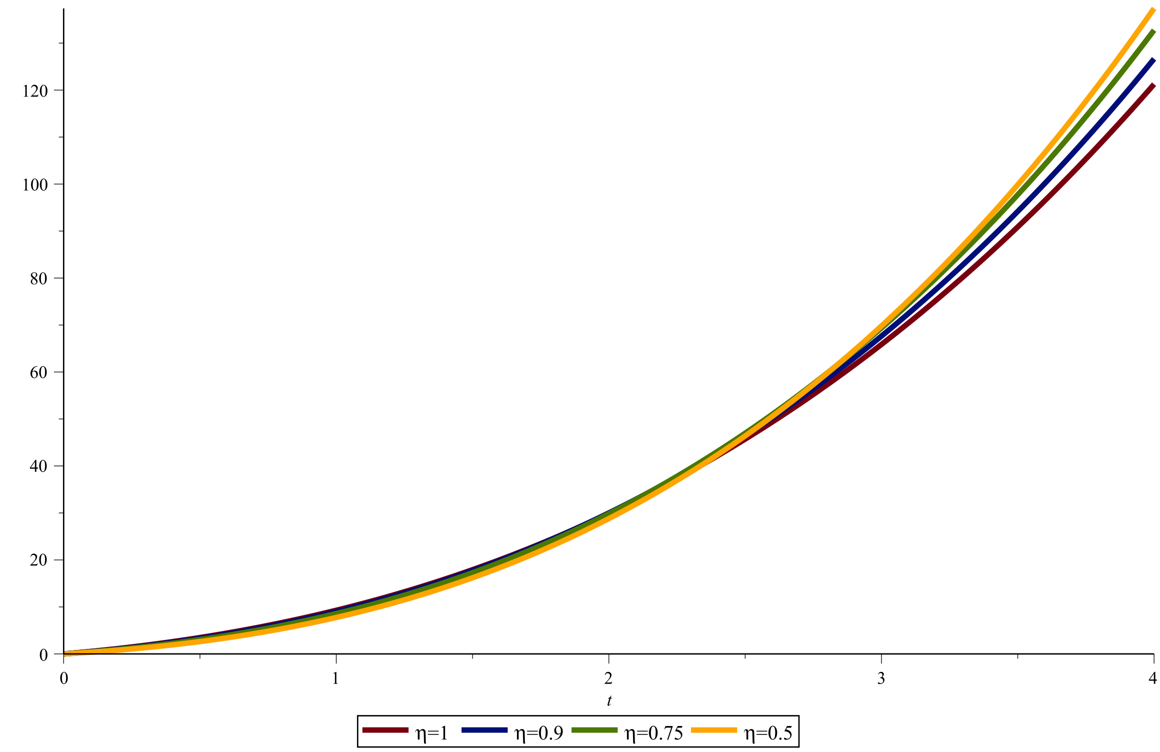

Fig. 2. The approximate solution of Example 5.2.

Table 2
(MAE) at various values of η.

η	(MAE)
0.9	9.107×10^{-3}
0.75	7.558×10^{-3}
0.5	3.608×10^{-15}

$$+\int_0^\tau \left[\left(\frac{c_0}{\Gamma(\eta+1)}t^\eta+\frac{2c_1}{\Gamma(\eta+2)}t^{\eta+1}-\frac{c_1}{\Gamma(\eta+1)}t^\eta+\frac{12c_2}{\Gamma(\eta+3)}t^{\eta+2}-\frac{6c_2}{\Gamma(\eta+2)}t^{\eta+1}+\frac{c_2}{\Gamma(\eta+1)}t^\eta\right)\right.$$
$$\left.-\frac{c_2}{2}t^4+\left(\frac{c_1}{3}-c_2\right)t^3+\left(\frac{c_0-c_1+c_2}{2}\right)t^2+\frac{2\Gamma\left(\frac{7}{2}\right)t^{\frac{5}{2}+\wp}}{\Gamma\left(\frac{7}{2}+\wp\right)}-2t^{\frac{5}{2}}\right]^2 dt\Bigg]d\tau.$$

Now, we define

$$\mathcal{V}(c_0,c_1,c_2,\dots,c_n)=\mathcal{J}[\mathscr{Y}]-\lambda\left[\mathscr{Y}(2)-2^{\frac{7}{2}}\right].$$

Differentiating $\mathcal{V}$ with regards to c_i and putting them equal to zero, then we get a system of $(n+1)$ algebraic equations with $(n+1)$ undetermined coefficients c_i.

Therefore the solution of the problem (5.7)–(5.11), now becomes ready by substituting the determined coefficients c_i, into Eq. (5.14).

Table 2 shows the maximum absolute errors(MAE) related to the problem (5.7)–(5.11) at different values of η.

Finally Fig. 2 represent the approximate solution of problem (5.7)–(5.11) at various values of η.

6. Conclusions

In the current study, an effective and straightforward technique is created for resolving many types of fractional variational problems with time delay, which are frequently utilized as models in engineering sciences. We reduce the problem to a set of algebraic equations that must be solved while tackling variational problems using the shifted Legendre functions and Laplace transform. The practicability of the devised method was examined on two numerical examples. It may be said that our approach is very powerful at both approximate and analytical solution investigating.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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