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Closure Operators on Graphs

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Abstract: The aim of this paper is to generate topological structure on the power set of vertices of digraphs using new definition which is G_m-closure operator on out-linked of digraphs. Properties of this topological structure are studied and several examples are given. Also we give some new generalizations of some definitions in digraphs to the some known definitions in topology which are R-open subgraph, α-open subgraph, pre-open subgraph, and β-open subgraph. Furthermore, we define and study the accuracy of these new generalizations on subgraps and paths.

Key words :Closure operators, dense set, regular open set, near open sets, accuracy.

INTRODUCTION

The use of topological ideas to explore various aspects of graph theory, and vice versa, is a fruitful area of research. There are links with other areas of mathematics, such as design theory and geometry, and increasingly with such areas as computer networks where symmetry is an important feature. This paper is part of an on-going project in which we seek to explore how standard facts about topological spaces in finite graphs can best be generalized to infinite graphs. The basic idea is that such structure can get topological closure spaces by using closure operators on graphs.

A *graph* (resp., *directed graph* or *digraph*) (R.J. Wilson, 1996), G=(V(G), E(G)) consists of a vertex set V(G) and an edge set E(G) of unordered (resp., ordered) pairs of elements of V(G). To avoid ambiguities, we assume that the vertex and edge sets are disjoint. We say that two vertices *v* and *w* of a graph (resp., digraph) G are *adjacent* if there is an edge of the form *vw* (rsep., $\vec{vw}$ or $\vec{wv}$) joining them, and the vertices *v* and *w* are then *incident* with such an edge. A *subgraph* (W.D. Wallis, 2007), of a graph G is a graph, each of whose vertices belong to V(G) and each of whose edges belong to E(G). The *degree* of a vertex *v* of G is the number of edges incident with *v*, and written deg(v). A vertex of degree zero is an *isolated* vertex. In digraph, the *out-degree* (J. Bondy and D.S. Murty, 1992), of a vertex *v* of G is the number of edges of the form $\vec{vw}$, and denoted by $D^+(v)$, similarly, the *in-degree* of a vertex v of G is the number of edges of the form $\vec{wv}$, and denoted by $D^-(v)$. A vertex of out-degree and in-degree are zero is an *isolated* vertex. A graph whose edge-set is empty is a *null* graph; we denote the null graph on n vertices by N_n. A *walk* (R. Diestel, 2005) is a 'way of getting from one vertex to another', and consists of a sequence of edges, one following after another. A walk in which no vertex appears more than once is called a *path*. For other notions or notations in topology not defined here we follow closely (R. Englking, 1989; S. Willard, 1970).

Closure Operators on Graphs:

In this section, we introduce and study the concepts of closure operators on digraphs, several known topological property on the obtained G_m-closure spaces are studies, and we introduce the concept of G_m-dense subgraphs in G_m-closure spaces.

Definition 2.1:

Let G=(V(G), E(G)) be a digraph, P(V(G)) its power set of all subgraphs of G and Cl_G:P(V(G))→P(V(G)) is a mapping associating with each subgraph H=(V(H), E(H)) a subgraph $Cl_G(V(H)) \subseteq V(G)$ called the closure subgraph of H such that: $Cl_G(V(H))=V(H) \cup \{v \in V(G)\backslash V(H); \vec{hv} \in E(G) \text{ for all } h \in V(H)\}$

The operation Cl_G is called graph closure operator and the pair (V(G), $\boldsymbol{C}_G$) is called G-closure space, where $\boldsymbol{C}_G$(V(G)) is the family of elements of Cl_G. The dual of the graph closure operator Cl_G is the graph interior operator Int_G:P(V(G))→P(V(G)) defined by $Int_G(V(H))=V(G)\backslash Cl_G(V(G)\backslash V(H))$ for all subgraph $H \subseteq G$. A family of elements of Int_G is called interior subgraph of H and denoted by $\boldsymbol{O}_G$(V(G)). Clear that (V(G), $\boldsymbol{O}_G$) is a topological space. Then the domain of Cl_G is equal to the domain of Int_G and also $Cl_G(V(H))=V(G)\backslash Int_G(V(G)\backslash V(H))$. A subgraph H of G-closure space (V(G), $\boldsymbol{C}_G$) is called closed subgraph if $Cl_G(V(H))=V(H)$. It is called open subgraph if its complement is closed subgraph, i.e., $Cl_G(V(G)\backslash V(H))=V(G)\backslash V(H)$, or equivalently $Int_G(V(H))=V(H)$.

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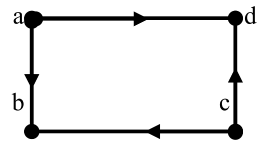
Example 2.1:
Let G=(V(G), E(G)) be a digraph such that:
V(G)={a, b, c, d},
E(G)={(a, b), (a, d), (c, b), (c, d)}.



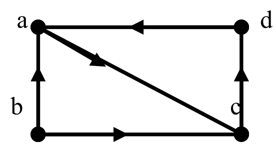
V(H)	Cl_G(V(H))	V(H)	Cl_G(V(H))
V(G)	V(G)	{a, d}	{a, b, d}
φ	φ	{b, c}	{b, c, d}
{a}	{a, b, d}	{b, d}	{b, d}
{b}	{b}	{c, d}	{ b, c, d}
{c}	{b, c, d}	{a, b, c}	V(G)
{d}	{d}	{a, b, d}	{a, b, d}
{a, b}	{a, b, d}	{a, c, d}	V(G)
{a, c}	V(G)	{b, c, d}	{b, c, d}

$\boldsymbol{C}_G$(V(G))={V(G), φ, {b}, {d}, {b, d}, {a, b, d}, {b, c, d}},
$\boldsymbol{O}_G$(V(G))={V(G), φ , {a}, {c}, {a, c}, {a, b, c}, {a, c, d}}.

We obtain a new definition to construct topological closure spaces from G-closure spaces by redefine graph closure operator on the resultant subgraphs as a domain of the graph closure operator and stop when the operator transfers each subgraph to itself.

Definition 2.2:
Let G=(V(G), E(G)) be a digraph and Cl_{Gm}:P(V(G))→P(V(G)) an operator such that:
(a) It is called G_m-closure operator if $Cl_{Gm}(V(H))=Cl_G(Cl_G(\ldots Cl_G(V(H))))$. m-times, for every subgraph $H\subseteq G$,
(b) it is called G_m-topological closure operator if $Cl_{Gm+1}(V(H)) =Cl_{Gm}(V(H))$ for all subgraph $H\subseteq G$.
The space (V(G), $\boldsymbol{C}_{Gm}$) is called G_m-closure space.

Example 2.2:
Let G=(V(G), E(G)) be a digraph such that:
V(G)={a, b, c, d},
E(G)={(a, c), (b, a) , (b, c) , (c, d) , (d, a)}.



V(H)	Cl_G(V(H))	Cl_{G2}(V(H))	V(H)	Cl_G(V(H))	Cl_{G2}(V(H))
V(G)	V(G)	V(G)	{a, d}	{a, c, d}	{a, c, d}
φ	φ	φ	{b, c}	V(G)	V(G)
{a}	{a, c}	{a, c, d}	{b, d}	V(G)	V(G)
{b}	{a, b, c}	V(G)	{c, d}	{a, c, d}	{a, c, d}
{c}	{c, d}	{a, c,d}	{a, b, c}	V(G)	V(G)
{d}	{a, d}	{a, c, d}	{a, b, d}	V(G)	V(G)
{a, b}	{a, b, c}	V(G)	{a, c, d}	{a, c, d}	{a, c, d}
{a, c}	{a, c, d}	{a, c, d}	{b, c, d}	V(G)	V(G)

$\boldsymbol{C}_{G2}$(V(G))={V(G), φ, {a, c, d}},
$\boldsymbol{O}_{G2}$(V(G))={V(G), φ, {b}}.

Proposition 2.1:
Let G=(V(G), E(G)) be a digraph, and (V(G), $\boldsymbol{C}_{Gm}$) be G_m-closure space. If H=(V(H), E(H)), K=(V(K), E(K)) are two subgraphs of G such that $H\subseteq K\subseteq G$, then $Cl_{Gm}(V(H))\subseteq Cl_{Gm}(V(K))$, and $Int_{Gm}(V(H))\subseteq Int_{Gm}(V(K))$.
Proof: Let $x\in Cl_{Gm}(V(H))$

$\Rightarrow x\in V(H)\cup \{v\in V(G)\backslash V(H); \overrightarrow{hv} \in E(G) \text{ for all } h\in V(H)\}$

$\Rightarrow x\in V(H) \text{ or } \{v\in V(G)\backslash V(H); \overrightarrow{hv} \in E(G) \text{ for all } h\in V(H)\}$

$\Rightarrow x\in V(H) \text{ or } \exists\ h\in V(H); \overrightarrow{hx} \in E(G), \text{ Since } H\subseteq K$

$\Rightarrow x\in V(K) \text{ or } \exists h\in V(K)\ \overrightarrow{hx} \in E(G)$

$\Rightarrow x\in V(K) \text{ or } \{v\in V(G)\backslash V(K); \overrightarrow{hv} \in E(G) \text{ for all } h\in V(K)\}$

$\Rightarrow x\in V(K)\cup \{v\in V(G)\backslash V(K); \overrightarrow{hv} \in E(G) \text{ for all } h\in V(K)\}$

$\Rightarrow x \in Cl_{Gm}(V(K))$. Hence $Cl_{Gm}(V(H)) \subseteq Cl_{Gm}(V(K))$.
Now, let $x \in Int_{Gm}(V(H))=V(G)\backslash Cl_{Gm}(V(G)\backslash V(H)) \Rightarrow x \notin Cl_{Gm}(V(G)\backslash V(H))$, since $V(H) \subseteq V(K) \Rightarrow V(G)\backslash V(K) \subseteq V(G)\backslash V(H) \Rightarrow Cl_{Gm}(V(G)\backslash V(K)) \subseteq Cl_{Gm}(V(G)\backslash V(H))$. So, $x \notin Cl_{Gm}(V(G)\backslash V(K)) \Rightarrow x \in V(G)\backslash Cl_{Gm}(V(G)\backslash V(K)) \Rightarrow x \in Int_{Gm}(V(K))$. Hence, $Int_{Gm}(V(H)) \subseteq Int_{Gm}(V(K))$.

Proposition 2.2:

Let G=(V(G), E(G)) be a digraph, $(V(G), \boldsymbol{C}_{Gm})$ be G_m-closure space, and H=(V(H), E(H)), K=(V(K), E(K)) are two subgraphs of G, then

(a) $Cl_{Gm}(V(H) \cup V(K))=Cl_{Gm}(V(H)) \cup Cl_{Gm}(V(K))$.

(b) $Int_{Gm}(V(H) \cap V(K))=Int_{Gm}(V(H)) \cap Int_{Gm}(V(K))$.

Proof: (a) Let $x \in Cl_{Gm}(V(H) \cup V(K))$

$\Leftrightarrow x \in (V(H) \cup V(K)) \cup \{v \in V(G)\backslash(V(H) \cup V(K)); \overrightarrow{gv} \in E(G)$ for all $g \in (V(H) \cup V(K))$

$\Leftrightarrow x \in (V(H) \cup V(K))$ or $x \in \{v \in V(G)\backslash(V(H) \cup V(K)); \overrightarrow{gv} \in E(G)$for all $g \in (V(H) \cup V(K))$

$\Leftrightarrow$ ($x \in (V(H)$ or $x \in V(K))$ or ($\exists g \in (V(H) \cup V(K)); \overrightarrow{xg} \in E(G))$

$\Leftrightarrow$ ($x \in V(H)$ or $x \in V(K))$ or ($\exists g \in V(H); \overrightarrow{xg} \in E(G)$ or ($\exists g \in V(K)); \overrightarrow{xg} \in E(G))$

$\Leftrightarrow$ ($x \in V(H)$ or $\exists g \in (V(H); \overrightarrow{xg} \in E(G)$ or ($x \in V(K)$ or $\exists g \in V(K); \overrightarrow{xg} \in E(G))$

$\Leftrightarrow x \in V(H)$ or $x \in \{v \in V(G)\backslash V(H); \overrightarrow{gv} \in E(G)$ for all $g \in V(H)\}$

or $x \in V(K)$ or $x \in \{v \in V(G)\backslash V(K); \overrightarrow{gv} \in E(G)$ for all $g \in V(K)\}$

$\Leftrightarrow x \in V(H) \cup \{v \in V(G)\backslash V(H); \overrightarrow{gv} \in E(G)$ for all $g \in V(H)\}$

or $x \in V(K) \cup \{v \in V(G)\backslash V(K); \overrightarrow{gv} \in E(G)$ for all $g \in V(K)\}$

$\Leftrightarrow x \in Cl_{Gm}(V(H))$ or $x \in Cl_{Gm}(V(K))$

$\Leftrightarrow x \in (Cl_{Gm}(V(H) \cup Cl_{Gm}(V(K))$.

(b) $Int_{Gm}(V(H) \cap V(K))=V(G)\backslash\ Cl_{Gm}(V(G)\backslash V(H) \cap V(K))$

$=V(G)\backslash\ Cl_{Gm}[(V(G)\backslash V(H)) \cup ((V(G)\backslash V(K))]$

$=V(G)\backslash[Cl_{Gm}(V(G)\backslash V(H)] \cup [Cl_{Gm}(V(G)\backslash V(K))]$

$=[V(G)\backslash\ Cl_{Gm}(V(G)\backslash V(H)] \cap [V(G)\backslash Cl_{Gm}(V(G)\backslash V(K))]$

$=Int_{Gm}(V(H)) \cap Int_{Gm}(V(K))$.

Proposition 2.3:

Let G=(V(G), E(G)) be a digraph, $(V(G), \boldsymbol{C}_{Gm})$ be G_m-closure space, and H=(V(H), E(H)), K=(V(K), E(K)) be two subgraphs of G, then

(a) $Cl_{Gm}(V(H) \cap V(K)) \subseteq Cl_{Gm}(V(H)) \cap Cl_{Gm}(V(K))$, and

(b) $Int_{Gm}(V(H)) \cup Int_{Gm}(V(K)) \subseteq Int_{Gm}(V(H) \cup V(K))$.

Proof: (a) Since $(V(H) \cap V(K)) \subseteq V(H)$ and $(V(H) \cap V(K)) \subseteq V(K)$

$\Rightarrow Cl_{Gm}(V(H) \cap V(K)) \subseteq Cl_{Gm}(V(H))$ and $Cl_{Gm}(V(H) \cap V(K)) \subseteq Cl_{Gm}(V(K))$

$\Rightarrow Cl_{Gm}(V(H) \cap V(K)) \subseteq Cl_{Gm}(V(H)) \cap Cl_{Gm}(V(K))$.

(b) Since $V(H) \subseteq (V(H) \cup V(K))$ and $V(K) \subseteq (V(H) \cup V(K))$

$\Rightarrow Int_{Gm}(V(H)) \subseteq Int_{Gm}(V(H) \cup V(K))$ and $Int_{Gm}(V(K)) \subseteq Int_{Gm}(V(H) \cup V(K))$

$\Rightarrow Int_{Gm}(V(H)) \cup Int_{Gm}(V(K)) \subseteq Int_{Gm}(V(H) \cup V(K))$.

Remark 2.1:

The converse of proposition (2.3) above need not be true in general, as the following example shows.

Example 2.3:

Consider the digraph on example (2.1)

(a) Let H=(V(H), E(H)); V(H)={a, b}, E(H)={(a, b)}

K=(V(K), E(K)); V(K)={b, c}, E(K)={(c, b)}

Then, $Cl_G(V(H)) \cap Cl_G(V(K))=\{a, b, d\} \cap \{b, c, d\}=\{b, d\}$
and $Cl_G(V(H) \cap V(K)=Cl_G(\{b\})=\{b\}$, but $\{b, d\} \not\subset \{b\}$.
(b) Let H=(V(H), E(H)); V(H)={a, b, d}, E(H)={(a, b), (a, d)}
K=(V(K), E(K)); V(K)={b, c, d}, E(K)={(c, b), (c, d)}
Then, $Int_G(V(H) \cup V(K))=Int_G(V(G))=V(G)$
and $Int_G(V(H)) \cup Int_G(V(K))=\{a\} \cup \{c\}=\{a, c\}$, but $V(G) \not\subset \{a, c\}$.

Proposition 2.4:
Let G=(V(G), E(G)) be a digraph, (V(G), $\boldsymbol{C}_{Gm}$) be G_m-closure space, and H=(V(H), E(H)) be a subgraphs of G, then $Cl_{Gm}(V(H))=V(H)$ if
(a) All vertices of H are isolated; or
(b) For all vertex $v \in V(H)$, we have $D^-(v) \geq 1$ and $D^+(v)=0$.
Proof: Clear.
Now, we introduce the definition of G_m-dense of subgraph in G_m-closure space as follows.

Definition 2.3:
Let G=(V(G),E(G)) be a digraph, and (V(G), $\boldsymbol{C}_{Gm}$) be G_m-closure space. A subgraph H=(V(H), E(H)) is called G_m-dense in G iff $Cl_{Gm}(V(H))=V(G)$.

Example 2.4:
In example (2.1) the set of all G-dense subgraph is
{{a, c}, {a, b, c}, {a, c, d}} since
$Cl_G(\{a, c\})=V(G)$, $Cl_G(\{a, b, c\})=V(G)$, $Cl_G(\{a, c, d\})=V(G)$.
In example (2.2) the set of all G_2-dense subgraph is
{{b}, {a, b}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {b, c, d}} since
$Cl_{G2}(\{b\})=V(G)$, $Cl_{G2}(\{a,b\})=V(G)$, $Cl_{G2}(\{b,c\})=V(G)$, $Cl_{G2}(\{b, d\})=V(G)$, $Cl_{G2}(\{a, b, c\})=V(G)$, $Cl_{G2}(\{a, b, d\})=V(G)$, $Cl_{G2}(\{b, c, d\})=V(G)$.
We give three conditions which are equivalent to saying a subgraph H=(V(H), E(H)) of a digraph G=(V(G), E(G)) is G_m-dense in the G_m-closure space (V(G), $\boldsymbol{C}_{Gm}$).

Theorem 2.1:
Let G=(V(G), E(G)) be a digraph, and H=(V(H), E(H)) be a subgraph of G_m-closure space (V(G), $\boldsymbol{C}_{Gm}$). Then the following four conditions are equivalent:
(a) The subgraph H is G_m-dense in G,
(b) If K=(V(K), E(K)) any closed subgraph of G and $H \subseteq K$, then K=G,
(c) For each vertex $v \in V(G)$, every open subgraph in G containing v has a nonempty intersection with H,
(d) $Int_{Gm}(V(G)\backslash V(H))=\varphi$.
Proof: (a)$\Rightarrow$(b). From (a), we know $Cl_{Gm}(V(H))=V(G)$. Now let K be a closed subgraph of G such that $H \subseteq K$. Since K is a closed subgraph, we have $V(G)=Cl_{Gm}(V(H)) \subseteq Cl_{Gm}(V(K))=V(K)$. Hence K=G.
(b)$\Rightarrow$(c). Let $v \in V(G)$, and O be a nonempty open subgraph in G containing v. Assume that $V(O) \cap V(H)=\varphi$, then $V(H) \subseteq V(G)\backslash V(O)$. The fact that $V(G)\backslash V(O)$ is a closed subgraph in G allows us to use (b) infer that $V(G)\backslash V(O)=V(G)$. But, on the other hand, $V(O) \neq \varphi$ implies $V(G)\backslash V(O) \neq V(G)$. This contradiction means that our assumption is false, so that $V(O) \cap V(H) \neq \varphi$.
(c)$\Rightarrow$(d). Assume $Int_{Gm}(V(G)\backslash V(H)) \neq \varphi$. Then $Int_{Gm}(V(G)\backslash V(H))$ is a nonempty set which is open subgraph. However, $(V(G)\backslash V(H)) \cap V(H)=\varphi$ and since $Int_{Gm}(V(G)\backslash V(H)) \subseteq V(G)\backslash V(H)$, we have $Int_{Gm}(V(G)\backslash V(H)) \cap V(H)=\varphi$. This contradiction (c) and means $Int_{Gm}(V(G)\backslash V(H))=\varphi$.
(d)$\Rightarrow$(a). By definition, $Cl_{Gm}(V(H))=V(G)\backslash Int_{Gm}(V(G)\backslash V(H))=V(G)\backslash\varphi= V(G)$. Hence H is G_m-dense in G.

(R, α, pre, β)-Open Subgraphs:
In this section, we introduce and study (R, α, pre, β)-open subgraphs, and we defined and study the accuracy (resp., i-accuracy, ij-accuracy); $i,j \in \{R, \alpha, pre, \beta\}$ of subgraphs and paths in G_m-closure spaces.
By a similar way of definitions of regular open set (M. Stone, 1937), α-open set (O. Njastad, 1956), pre-open set (A.S. Mashhour, *et al*., 1982) and β-open set (M.E. Abd El-Monsef, *et al*., 1983) (=semi-pre-open set (D. Andrijevic, 1986)), we introduce the following definitions:

Definition 3.1:
Let G=(V(G), E(G)) be a digraph and (V(G), C_{Gm}) its G_m-closure space.
(a)An open subgraph H=(V(H), E(H)) in (V(G), C_{Gm}) is called regular open subgraph (briefly R-open subgraph) if $V(H)=Int_{Gm}(Cl_{Gm}(V(H)))$, the complement of R-open subgraph is called R-closed subgraph. The R-closure

subgraph of H is $Cl^{R}_{Gm}(V(H))=\cap\{V(F);\ V(F)$ is R-closed subgraph, $V(H)\subseteq V(F)\}$, and $Int^{R}_{Gm}(V(H))=V(G)\setminus Cl^{R}_{Gm}(V(G)\setminus V(H))$.

The family of all R-open subgraph (resp., R-closed subgraph) of (V(G), C_{Gm}) is denoted by $O^{R}_{Gm}(V(G))$ (resp., $C^{R}_{Gm}(V(G))$).

A subgraph H=(V(H), E(H)) in (V(G), C_{Gm}) is called

(b) α-open subgraph if $V(H)\subseteq Int_{Gm}(Cl_{Gm}(Int_{Gm}(V(H))))$, the complement of α-open subgraph is called α-closed subgraph. The α-closure subgraph of H is $Cl^{\alpha}_{Gm}(V(H))=\cap\{V(F);\ V(F)$ is α-closed subgraph, $V(H)\subseteq V(F)\}$, and $Int^{\alpha}_{Gm}(V(H))=V(G)\setminus Cl^{\alpha}_{Gm}(V(G)\setminus V(H))$.

The family of all α-open subgraph (resp., α-closed subgraph) of (V(G), C_{Gm}) is denoted by $O^{\alpha}_{Gm}(V(G))$ (resp., $C^{\alpha}_{Gm}(V(G))$). Clear that (V(G), O^{α}_{Gm}) is topological space.

(c) pre-open subgraph if $V(H)\subseteq Int_{Gm}(Cl_{Gm}(V(H)))$, the complement of pre-open subgraph is called pre-closed subgraph. The pre-closure subgraph of H is $Cl^{Pre}_{Gm}(V(H))=\cap\{V(F);\ V(F)$ is pre-closed subgraph, $V(H)\subseteq V(F)\}$, and $Int^{Pre}_{Gm}(V(H))=V(G)\setminus Cl^{Pre}_{Gm}(V(G)\setminus V(H))$.

The family of all pre-open subgraph (resp., pre-closed subgraph) of (V(G), C_{Gm}) is denoted by $O^{Pre}_{Gm}(V(G))$ (resp., $C^{Pre}_{Gm}(V(G))$).

(d) β-open subgraph if $V(H)\subseteq Cl_{Gm}(Int_{Gm}(Cl_{Gm}(V(H))))$, the complement of β-open subgraph is called β-closed subgraph. The β-closure subgraph of H is $Cl^{\beta}_{Gm}(V(H))=\cap\{V(F);\ V(F)$ is β-closed subgraph, $V(H)\subseteq V(F)\}$, and $Int^{\beta}_{Gm}(V(H))=V(G)\setminus Cl^{\beta}_{Gm}(V(G)\setminus V(H))$.

The family of all β-open subgraph (resp., β-closed subgraph) of (V(G), C_{Gm}) is denoted by $O^{\beta}_{Gm}(V(G))$ (resp., $C^{\beta}_{Gm}(V(G))$).

Example 3.1:

Consider the digraph in example (2.1), then we have

$O^{R}_{Gm}(V(G))=\{V(G), \varphi, \{a\}, \{c\}\}$,

$C^{R}_{Gm}(V(G))=\{V(G), \varphi, \{a, b, d\}, \{b, c, d\}\}$,

$O^{\alpha}_{Gm}(V(G))=\{V(G), \varphi, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$,

$C^{\alpha}_{Gm}(V(G))=\{V(G), \varphi, \{b\}, \{d\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}$,

$O^{Pre}_{Gm}(V(G))=\{V(G), \varphi, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$,

$C^{Pre}_{Gm}(V(G))=\{V(G), \varphi, \{b\}, \{d\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}$,

$O^{\beta}_{Gm}(V(G))=\{V(G), \varphi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$, and

$C^{\beta}_{Gm}(V(G))=\{V(G), \varphi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{b, c, d\}\}$.

Example 3.2:

Consider the digraph in example (2.2), then we have

$O^{R}_{Gm}(V(G))=\{V(G), \varphi\}$,

$C^{R}_{Gm}(V(G))=\{V(G), \varphi\}$,

$O^{\alpha}_{Gm}(V(G))=\{V(G), \varphi, \{b\}, \{a, b\}, \{b,c\}, \{b,d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$,

$C^{\alpha}_{Gm}(V(G))=\{V(G), \varphi, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, c, d\}\}$,

$O^{Pre}_{Gm}(V(G))=\{V(G), \varphi, \{b\}, \{a,b\}, \{b,c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$,

C_{Gm}^{Pre} (V(G))={V(G), φ, {a}, {c}, {d}, {a, c}, {a, d}, {c, d}, {a, c, d}},

O_{Gm}^{β} (V(G))={V(G), φ,{b},{a,b},{b,c},{b, d},{a, b, c},{a, b, d},{b, c, d}}, and

C_{Gm}^{β} (V(G))={V(G), φ, {a}, {c}, {d}, {a, c}, {a, d}, {c, d}, {a, c, d}}.

Proposition 3.1:

Let G=(V(G), E(G)) be a digraph, and (V(G), C_{Gm}) be G_m-closure space, then the following statements are true.

(a) O_{Gm}^{R} (V(G)) ⊆ O_{Gm}(V(G)) ⊆ O_{Gm}^{α} (V(G)) ⊆ O_{Gm}^{Pre} (V(G) ⊆ O_{Gm}^{β} (V(G)).

(b) C_{Gm}^{R} (V(G)) ⊆ C_{Gm}(V(G)) ⊆ C_{Gm}^{α} (V(G)) ⊆ C_{Gm}^{Pre} (V(G)) ⊆ C_{Gm}^{β} (V(G)).

Proof: (a)

(i) By definition of R-open subgraph, we have O_{Gm}^{R} (V(G)) ⊆ O_{Gm}(V(G)).

(ii) Let H=(V(H), E(H)) be open subgraph, implies V(H)=Int_{Gm}(V(H)), but Int_{Gm}(V(H)) ⊆ Cl_{Gm}(Int_{Gm}(V(H))), so V(H) ⊆ Cl_{Gm}(Int_{Gm}(V(H))) and Int_{Gm}(V(H)) ⊆ Int_{Gm}(Cl_{Gm}(Int_{Gm}(V(H)))). Therefore V(H) ⊆ Int_{Gm}(Cl_{Gm}(Int_{Gm}(V(H)))), and H is α-open subgraph.

(iii) Let H=(V(H), E(H)) be α-open subgraph, then V(H) ⊆ Int_{Gm}(Cl_{Gm}(Int_{Gm}(V(H)))), since Int_{Gm}(V(H)) ⊆ V(H), this implies Int_{Gm}(Cl_{Gm}(Int_{Gm}(V(H))) ⊆ Int_{Gm}(Cl_{Gm}(V(H))). Hence V(H) ⊆ Int_{Gm}(Cl_{Gm}(V(H))). Therefore H is pre-open subgraph.

(iv) Let H=(V(H), E(H)) be pre-open subgraph, then V(H) ⊆ Int_{Gm}(Cl_{Gm}(V(H))). This implies but Cl_{Gm}(V(H)) ⊆ Cl_{Gm}(Int_{Gm}(Cl_{Gm}(V(H))))), implies V(H) ⊆ Cl_{Gm}(Int_{Gm}(Cl_{Gm}(V(H)))). Which means H is β-open subgraph.

(b) Clear.

Remark 3.1:

In general the converse of proposition (3.1) above need not be true as the following example shows.

Example 3.3:

(a) In example (3.1) the subgraph H=(V(H), E(H)); V(H)={a, b, d}, E(H)={(a, b), (a, d)} is open subgraph but not R-open subgraph.

(b) In example (3.2) the subgraph H=(V(H), E(H)); V(H)={a, b, c}, E(H)={(a, c), (b, a), (b, c)} is α-open subgraph but not open subgraph.

(c) Consider the digraph G=(V(G), E(G)) where

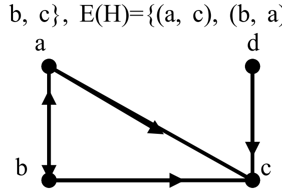
C_G(V(G))={V(G), φ, {c}, {c, d}, {a, b, c}},

O(V(G))={V(G), φ, {d}, {a, b}, {a, b, d}},

O_{G}^{α} (V(G))={V(G), φ,{d},{a, b},{a, b, d}} and

O_{G}^{Pre} (V(G))={V(G), φ,{a},{b},{d},{a, b},{a, d},{b, d},{a, b, d},

{a, c, d}, {b, c, d}}.



The subgraph H=(V(H), E(H)); V(H)={a, c, d}, E(H)={(a, b), (a, c), (d, c)} is pre-open subgraph but not α-open subgraph.

(d) In example (3.1) the subgraph H=(V(H), E(H)); V(H)={a, d}, E(H)={(a, d)} is β-open subgraph but not pre-open subgraph.

If the digraph represente the world and the vertices represente the countries in the world and the economy of the country figure on trade. If the product of the country is increasing (Int_{Gm}) and the importation is decreasing (Cl_{Gm}), then the country has strong economy (accuracy) and vice versa. So if there is a link which conveys product from country a to country b the valuation (accuracy= μ_{Gm}(V(H))) where H is subgraph (country or countries) in the digraph G (world), corresponds to the rate of partnership economy of the country or countries in the world. Hence we introduce the definitions of accuracy, i-accuracy, and ij-accuracy of subgraph H; i,j ∈ {R, α, pre, β}.

Definition 3.2:

Let G=(V(G), E(G)) be a digraph, (V(G), $\boldsymbol{C}_{Gm}$) be G_m-closure space, and H=(V(H), E(H)) be a subgraphs of G=(V(G), E(G)), then

The accuracy of H is defined by $\mu_{Gm}(V(H)))=\dfrac{|\,Int_{Gm}(V(H))\,|}{|\,Cl_{Gm}(V(H))\,|}$.

The i-accuracy of H is defined by $\mu^{i}_{Gm}(V(H)) = \frac{|Int^{i}_{Gm}(V(H))|}{|Cl^{i}_{Gm}(V(H))|}$ and

$i \in \{R, \alpha, pre, \beta\}$.

The ij-accuracy of H is defined by $\mu^{ij}_{Gm}(V(H)) = \frac{|Int^{i}_{Gm}(V(H))|}{|Cl^{j}_{Gm}(V(H))|}$ and

$i,j \in \{R, \alpha, pre, \beta\}$.

We introduce an example to illustrate and clarifies the accuracy and i-accuracy $i \in \{R, \alpha, pre, \beta\}$ for any subgraph H in a digraph G as follows.

Example 3.4:

Consider the digraph in examples (2.1) and (3.1). The accuracy (resp., i-accuracy) of any subgraph is:

subgrapg	μ^{R}_{Gm} (H)	μ_{Gm} (H)	μ^{α}_{Gm} (H)	μ^{Pre}_{Gm} (H)	μ^{β}_{Gm} (H)
V(G)	1	1	1	1	1
φ	0	0	0	0	0
{a}	1/3	1/3	1/3	1/3	1
{b}	0	0	0	0	0
{c}	1/3	1/3	1/3	1/3	1
{d}	0	0	0	0	0
{a, b}	0	1/3	1/3	1/3	1
{a, c}	0	1/2	1/2	1/2	1/2
{a, d}	0	1/3	1/3	1/3	1
{b, c}	0	1/3	1/3	1/3	1
{b, d}	0	0	0	0	0
{c, d}	0	1/3	1/3	1/3	1
{a, b, c}	0	3/4	3/4	3/4	3/4
{a, b, d}	0	1/3	1/3	1/3	1
{a, c, d}	0	3/4	3/4	3/4	3/4
{b, c, d}	0	1/3	1/3	1/3	1

Proposition 3.2:

Let G=(V(G), E(G)) be a digraph, (V(G), $\boldsymbol{C}_{Gm}$) be G_m-closure space, and H=(V(H), E(H)) a subgraph of G, then $\mu^{R}_{Gm}(V(H)) \le \mu_{Gm}(V(H)) \le \mu^{\alpha}_{Gm}(V(H)) \le \mu^{Pre}_{Gm}(V(H)) \le \mu^{\beta}_{Gm}(V(H))$.

Proof: It is clear from proposition (3.1).

Proposition 3.3:

Let G=(V(G), E(G)) be a digraph, (V(G), $\boldsymbol{C}_{Gm}$) be G_m-closure space, and N_1 be a null subgraph of G containing one vertex, then $\mu_{Gm}(V(N_1))=1$

Proof: Let N_1 be a null subgraph of G such that $V(N_1)=\{v\}$, $E(N_1)=\varphi$. By proposition (2.4(i)) $Cl_{Gm}(V(N_1))=V(N_1)=\{v\}$, and $Int_{Gm}(V(N_1)) = V(G)\backslash Cl_{Gm}(V(G)\backslash V(N_1))=V(G)\backslash Cl_{Gm}(V(G)\backslash\{v\})$. Since v is an isolated vertex, then $D^{+}(v)=D^{-}(v)=0$, so $Cl_{Gm}(V(G)\backslash\{v\})=V(G)\backslash\{v\}$, and $Int_{Gm}(V(N_1))=V(G)\backslash(V(G)\backslash\{v\})=\{v\}=V(N_1)$. Hence $\mu_{Gm}(V(N_1))=1$.

Corollary 3.1:

Let G=(V(G), E(G)) be a digraph, (V(G), $\boldsymbol{C}_{Gm}$) be G_m-closure space, and N_n be a null subgraph of G such that $|V(N_n)| = n \ge 1$, then

$\mu^{R}_{Gm}(V(N_n))= \mu_{Gm}(V(N_n))= \mu^{\alpha}_{Gm}(V(N_n))= \mu^{Pre}_{Gm}(V(N_n))= \mu^{\beta}_{Gm}(V(N_n))=1$.

We introduce an example to illustrate and clarifies the ij-accuracy $i, j \in \{R, \alpha, pre, \beta\}$ for any subgraph H in a digraph G.

Example 3.5:

Consider the digraph in examples (2.1) and (3.1). The ij-accuracy; $i,j \in \{R, O, \alpha, pre, \beta\}$ where $O \in \boldsymbol{O}(V(G))$ of any subgraph is:

subgraph	μ_{Gm}^{RO}	$\mu_{Gm}^{R\alpha}$	$\mu_{Gm}^{R\,Pre}$	$\mu_{Gm}^{R\beta}$	μ_{Gm}^{OR}	$\mu_{Gm}^{O\alpha}$	$\mu_{Gm}^{O\,Pre}$	$\mu_{Gm}^{O\beta}$	$\mu_{Gm}^{\alpha R}$	$\mu_{Gm}^{\alpha O}$
V(G)	1	1	1	1	1	1	1	1	1	1
φ	0	0	0	0	0	0	0	0	0	0
{a}	1/3	1/3	1/3	1	1/3	1/3	1/3	1	1/3	1/3
{b}	0	0	0	0	0	0	0	0	0	0
{c}	1/3	1/3	1/3	1	1/3	1/3	1/3	1	1/3	1/3
{d}	0	0	0	0	0	0	0	0	0	0
{a, b}	1/3	1/3	1/3	1/2	1/3	1/3	1/3	1/2	1/3	1/3
{a, c}	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2
{a, d}	1/3	1/3	1/3	1/2	1/3	1/3	1/3	1/2	1/3	1/3
{b, c}	1/3	1/3	1/3	1/2	1/3	1/3	1/3	1/2	1/3	1/3
{b, d}	0	0	0	0	0	0	0	0	0	0
{c, d}	1/3	1/3	1/3	1/2	1/3	1/3	1/3	1/2	1/3	1/3
{a, b, c}	1/2	1/2	1/2	1/2	3/4	3/4	3/4	3/4	3/4	3/4
{a, b, d}	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3
{a, c, d}	1/2	1/2	1/2	1/2	3/4	1/2	1/2	3/4	3/4	3/4
{b, c, d}	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3

subgarph	$\mu_{Gm}^{\alpha\,Pre}$	$\mu_{Gm}^{\alpha\beta}$	μ_{Gm}^{PreR}	μ_{Gm}^{PreO}	$\mu_{Gm}^{Pre\alpha}$	$\mu_{Gm}^{Pre\beta}$	$\mu_{Gm}^{\beta R}$	$\mu_{Gm}^{\beta O}$	$\mu_{Gm}^{\beta\alpha}$	$\mu_{Gm}^{\beta\,Pre}$
V(G)	1	1	1	1	1	1	1	1	1	1
φ	0	0	0	0	0	0	0	0	0	0
{a}	1/3	1	1/3	1/3	1/3	1	1/3	1/3	1/3	1/3
{b}	0	0	0	0	0	0	0	0	0	0
{c}	1/3	1	1/3	1/3	1/3	1	1/3	1/3	1/3	1/3
{d}	0	0	0	0	0	0	0	0	0	0
{a, b}	1/3	1/2	1/3	1/3	1/3	1/2	2/3	2/3	2/3	2/3
{a, c}	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2
{a, d}	1/3	1/2	1/3	1/3	1/3	1/2	2/3	2/3	2/3	2/3
{b, c}	1/3	1/2	1/3	1/3	1/3	1/2	2/3	2/3	2/3	2/3
{b, d}	0	0	0	0	0	0	0	0	0	0
{c, d}	1/3	1/2	1/3	1/3	1/3	1/2	2/3	2/3	2/3	2/3
{a, b,c}	3/4	3/4	3/4	3/4	3/4	3/4	3/4	3/4	3/4	3/4
{a, b,d}	1/3	1/3	1/3	1/3	1/3	1/3	1	1	1	1
{a, c,d}	3/4	1/2	3/4	3/4	1/2	1/2	3/4	3/4	3/4	3/4
{b, c,d}	1/3	1/3	1/3	1/3	1/3	1/3	1	1	1	1

Remark 3.2:

From examples (3.4) and (3.5) above we have:

(a) $\mu_{Gm}^{ij}(V(H)) \leq \mu_{Gm}^{i\beta}(V(H)),\ \mu_{Gm}^{\beta i}(V(H)),\ \mu_{Gm}^{Oi}(V(H));\ i,j \in \{R, O, \alpha, pre, \beta\}$.

(b) $\mu_{Gm}^{Pre\alpha}(V(H)) \leq \mu_{Gm}^{PreO}(V(H))\ ,\ \mu_{Gm}^{PreR}(V(H)) \leq \mu_{Gm}^{OR}(V(H))$.

Remark 3.3:

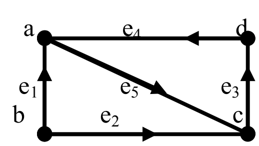
Let G=(V(G), E(G)) be a digraph, (V(G), $\boldsymbol{C}_{Gm}$) be G_m-closure space, and P be a path of G such that V(P) is a set of vertices of P, then

(a) The G_m-interior of P is $Int_{Gm}(V(P))=V(G)\backslash Cl_{Gm}(V(G)\backslash V(P))$.

(b) The G_m-closure of P is $Cl_{Gm}(V(P))=V(G)\backslash Int_{Gm}(V(G)\backslash V(P))$.

Example 3.6:

Consider the digraph in example (2.2)



Take the paths

$P_1=be_1ae_5c$; $V(P_1)=\{b, a, c\}$,

$P_2=ae_5ce_3d$; $V(P_2)=\{a, c, d\}$,

$P_3=ce_3d$; $V(P_3)=\{c, d\}$,

$P_1 \cap P_2=ae_5c$; $V(P_1 \cap P_2)=\{a, c\}$, and

$P_1 \cup P_3=be_1ae_5ce_3d$; $V(P_1 \cup P_3)=\{b, a, c, d\}$.

Paths	$\mu_{Gm}^{R}(P)$	$\mu_{Gm}(P)$	$\mu_{Gm}^{\alpha}(P)$	$\mu_{Gm}^{Pre}(P)$	$\mu_{Gm}^{\beta}(P)$
P_1	0	1/4	3/4	3/4	3/4
P_2	0	0	0	0	0
P_3	0	0	0	0	0
$P_1 \cap P_2$	0	0	0	0	0
$P_1 \cup P_2$	1	1	1	1	1

It is clear that

$\mu_{G2}(P_1)+\mu_{G2}(P_2)\leq\mu_{G2}(P_1\cup P_2)$,

$\mu^{i}_{G2}(P_2)+\mu^{i}_{G2}(P_2)\leq\mu^{i}_{G2}(P_1\cup P_2)$; $i\in\{R, \alpha, pre, \beta\}$,

$\mu_{G2}(P_1\cap P_2)\leq\mu_{G2}(P_1)+\mu_{G2}(P_2)$, and

$\mu^{i}_{G2}(P_1\cap P_2)\leq\mu^{i}_{G2}(P_1)+\mu^{i}_{G2}(P_2)$; $i\in\{R, \alpha, pre, \beta\}$.

Remark 3.4:

(a) The accuracy (resp., i-accuracy) of the union of two paths greatest or equal the sum of accuracy (resp., i-accuracy) of them; $i\in\{R, \alpha, pre, \beta\}$.

(b) The accuracy (resp., i-accuracy) of the intersection of two paths smallest or equal the sum of accuracy (resp., i-accuracy) of them; $i\in\{R, \alpha, pre, \beta\}$.

Remark 3.5:

Let G=(V(G), E(G)) be a digraph and (V(G), $\boldsymbol{C}_{Gm}$) its G_m-closure space,. If P_1, P_2 are two paths of G, then

(a) $Cl_{Gm}(P_1\cup P_2)=Cl_{Gm}(P_1)\cup Cl_{Gm}(P_2)$,

(b) $Cl_{Gm}(P_1\cap P_2)\subseteq Cl_{Gm}(P_1)\cap Cl_{Gm}(P_2)$,

(c) $Int_{Gm}(P_1\cap P_2)=Int_{Gm}(P_1)\cap Int_{Gm}(P_2)$, and

(d) $Int_{Gm}(P_1)\cup Int_{Gm}(P_2)\subseteq Int_{Gm}(P_1\cup P_2)$.

Definition 3.3:

Let G=(V(G), E(G)) be a digraph, (V(G), $\boldsymbol{C}_{Gm}$) be G_m-closure space and P a path of G such that V(P) is a set of vertices of P. The G_m-boundary of P is defined by: $Bd_{Gm}(V(P))=Cl_{Gm}(V(P))\cap Cl_{Gm}(V(G)\backslash V(P))$.

Proposition 3.4:

Let G=(V(G), E(G)) be a digraph, (V(G), $\boldsymbol{C}_{Gm}$) be G_m-closure space and P a path of G, then

(a) $Bd_{Gm}(V(P))=Bd_{Gm}(V(G)\backslash V(P))$.

(b) $Cl_{Gm}(V(P))=V(P)\cup Bd_{Gm}(V(P))$.

(c) $Int_{Gm}(V(P))=V(P)\backslash Bd_{Gm}(V(P))$.

Proof: Clear.

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