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Balqees K. Mahmoud and Yousif Y. Yousif

Feeble Hausdorff Spaces in Alpha- Topological Spaces Using Graph

Balqees K. Mahmoud [a] and Yousif Y. Yousif [b]

Department Of Mathematics, College Of Education For Pure Sciences(Ibn Al-Haitham), University of Baghdad, Baghdad , Iraq.

[a] Corresponding author: balqees.khaleel1003a@ihcoedu.uobaghdad.edu.iq
[b] yousif.y.y@ihcoedu.uobaghdad.edu.iq

Abstract. This paper introduces the definition of feebly Hausdorff spaces in special α-topological spaces constructed by elements of graph, and study some relationships with another concepts.

Keywords. feebly Hausdorff spaces, topologized graph.

INTRODUCTION

The concept of α-topology was introduced after the study and generalization of α-open sets (α-closed sets) by many mathematicians [1-8]. Separation axioms are one of the important concepts in topology. Many researchers used the term of generalized closed set to define some types of weakly separation axioms [2,4,10]. Hausdorff axiom is an interesting axiom which was useful in this search.
Graph is became an essential part of combinatorial applications before many years, and many sciences are interested by the uses of graphs in multiple aspects of mathematics knowledge [3]. Some Graphic concepts are defined in α-topological spaces [6,7]
The concept of feebly open sets (its complement feebly closed sets) introduced in topological spaces which are precisely associated with semi-open sets [5]. Also, feebly Hausdorff spaces are defined in [9].
In this paper, we introduce the concept of feebly Hausdorff spaces in α-topological spaces and its application in topologized graph. Furthermore, we study some specific properties and theorems with counter example.

PRELIMINARIES AND BASIC DEFINITIONS

In this section, we recall some definitions, and theorems that we need it to complete this search. We consider the non-empty set $X = V_G \cup E_G$, where V_G, E_G are the set of vertices, the set of edges respectively of any graph G, and define a topology τ on X. For any subset A of X, the interior (the closure) of A in X denoted as Int(A) (Cl(A)) with respect to τ. If A satisfied $A \subseteq Int(Cl(Int(A)))$ for all $A \subseteq X$, then A is called α-open set [8] while its complement is α-closed set or satisfy $Cl(Int(Cl(A))) \subseteq A$. In general, every open set is α-open, but the reverse is not true in general. The set of all α-open sets formed an (α-topology) denoted by τ_α, and (X, τ_α) is an α-topological space. Furthermore, the interior (the closure) of A in X with respect to τ_α is denoted by α-Int(A) = ∪ { B | B ⊆ A, B is α-open} (α-Cl(A) = ∩ { B | A ⊆ B, B is α-closed}). In addition, we define the surrounding set of x as the intersection of all open sets contained x which denoted by x°, and define the α-Cl(x) as the intersection of all α-closed sets contained x, where Cl(x) = ∩ { B | x ∈ B, B is closed}.

Separation axioms (T_0, T_1, T_2) are special properties for some spaces which satisfied there conditions. Sequentially, the relationship between open sets and α-open set resulted α-separation axioms (α-T_0, α-T_1, α-T_2), so

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every T_i-space (where i = 0, 1, 2) is α-T_i space. Hausdorff space is T_2space, which mean that any two distinct points can be separated by two disjoint open sets (their intersection is empty), so every Hausdorff space is α-Hausdorff space.

Definition 1 [6]. Let X be an α-topological space ,and $S \subseteq X$, the intersection of all α-open sets contained S is called α-surrounding set of S, and denoted by $S^{\alpha\diamond}$.

Definition 2 [7]. Let X be an α-connected topological space, then X is an α-prepath denoted as P_α if $|X| \leq 2$ with any α-topology except the indiscrete one, otherwise X satisfies the conditions of theorem (1) in [8] when $|X| \geq 3$.

Corollary 1 [7]: An α-topological space X is a locally α-connected if and only if each component of each α-open set in X is α-open.

Definition 3[7]. Let X be an α-connected topological space, an α-prepath is called an α-path if it is locally α-connected. When we have an α-prepath say ab, we can express it by ab-α-prepath, and it is an ab-α-path if it is locally α-connected.

Definition 4 [7]: An α-prepath is said to be α-bounded, if the compatible total orders are α-bounded. That means, it is an α-prepath with two terminal points.

Theorem 1 [7]. If P_α is an α-bounded prepath, then P_α is α-compact if and only if it is locally α-connected.

Definition 5 [7]. An α-topological space X is said to be α-topologized graph if :

• Any singleton is either α-open or α-closed.

• X has at most two α-boundary points, for all $x \in X$.

Definition 6 [9]. If X is a topological space, then X is feebly Hausdorff if for any two points x, y, $x \neq y$ there exist open sets U_x, U_y such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y \subseteq x^{\diamond}$.

Theorem 2 [9]. If X is a feebly Hausdorff topologized graph, and a, b are vertices in X with P an ab-path, then P is closed in X.

FEEBLE HAUSDORFF SPACES IN ALPHA- TOPOLOGICAL SPACES

In this section, we introduce some new concepts in α-topological space with some properties, and investigate some relationships between them with counter example.

Definition 7. Let X be an α-topological space ,and $x \in X$, the intersection of all α-open sets contained x is called α-surrounding set of x, and denoted by $x^{\alpha\diamond}$.

Definition 8. If X is an α-topological space, then X is α-feebly Hausdorff if for any two points x, y, $x \neq y$ there exist α-open sets U_x, U_y such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y \subseteq x^{\alpha\diamond}$.

Definition 9. Let X be an α-topological space, $S \subseteq X$, the intersection of $S^{\alpha\diamond}$ and $(X \setminus S)^{\alpha\diamond}$ is called the α-envelope of S and denoted as α-ρ(S).

Definition 10. Let X be an α-topological space, for any vertex $v \in X$, γ(v) is the set of all edges that incident on v.

Definition 11. An α-topological space X is said to be finitely adjacent, if $x^{\alpha\diamond} \cap y^{\alpha\diamond}$ is finite for any distinct points x and y. Furthermore, X is uniquely adjacent if the intersection contains mostly one point, and it is finitely incident if α-Cl(x) is finite for any $x \in X$.

The next theorem proved by the same idea of theorem (1,2) in [9].

Theorem 3. If X is an α-feebly Hausdorff topologized graph, and a, b are vertices in X with P_α an ab- α-path, then P_α is α-closed in X.

Proof: Assume that P_α is not α-closed, that means, $X \setminus P_\alpha$ is not α-open. So there is some points $v \in X \setminus P_\alpha$ such that v has non-empty intersection with P_α for every neighbourhood of v. Furthermore, v must be a vertex, because any singleton containing an edge is a neighbourhood of itself.

Since X is an α-feebly Hausdorff, for any vertex w in P_α there exist α-open sets U_W, N_W such that $w \in U_W$, $v \in N_W$, and $(U_W \cap N_W) \subseteq \rho(v) = \gamma(v)$. The set $\{U_w\}_{w \in P_\alpha \cap V}$ is an α-open cover of P_α, since every edge in P_α must be incident with some vertex in P_α, or else, the singleton would be both α-clopen in P_α. Furthermore, the α-bounded prepath P_α is α-compact by (2.6), so there exists a finite subset W of $V_X \cap P_\alpha = V_{P_\alpha}$ such that $\{U_w\}_{w \in W}$ is a finite subcover of P_α.

Now let $N = \bigcap_{w \in W} N_w$ be the intersection of α-open sets that contain v, so N is α-open. Therefore, N is an α-open neighbourhood of v, and we have that $(N \cap P_\alpha) \neq \emptyset$. But $(N \cap P_\alpha) \subseteq N \cap (\bigcup_{w \in W} U_w) = (\bigcap_{w \in W} N_w) \cap (\bigcup_{w \in W} U_w) \subseteq (\bigcup_{w \in W} N_w) \cap (\bigcup_{w \in W} U_w) = \bigcup_{w \in W}(N_w \cap U_w) \subseteq \gamma(v)$, so we obtain that N ∩

$P_\alpha$contains an edge e which incident with v. Hence e is incident with at least three vertices, two of them are a, b in P_α, and v $\notin P_\alpha$, that is contradiction with X is an α-topologized graph.

The following corollary shows that α-feebly Hausdorff property is heretible.

Corollary 3.7. Let X be an α-topological space, and S be any α-subspace of X. If X is α-feebly Hausdorff, then so is S.

Proof: If X is α-feebly Hausdorff and S is α-subspace of X, then for any two distinct points x, y, there exists α-open sets U_x, U_y of x, y such that $U_x \cap U_y \subseteq x^{\alpha\diamond}$. Hence $V_x = S \cap U_x$ and $V_y = S \cap U_y$ are disjoint α-open sets of x and y in S such that $V_x \cap V_y \subseteq x^{\alpha\diamond}$, so S is α-feebly Hausdorff.

The next important theorem shows the transitive property in α-feebly Hausdorff spaces.

Theorem 3.8. If X is an α-feebly Hausdorff topologized graph, a, b, c are three vertices of X, and P_α, Q_α are ab-α-path, bc-α-path respectively in X. Then an ac-α-path contained in $P_\alpha \cup Q_\alpha$.

Proof: Suppose that P_α and Q_α have two total orders $\leq_P$, $\leq_Q$ respectively such that a, b are minimums in P_α, Q_α respected to their total orders.

Let z = inf $(P_\alpha \cap Q_\alpha)$ with respect to $\leq_P$. So $z \in P_\alpha$because $b \in (P_\alpha \cap Q_\alpha)$ and have an order-complete $\leq_P$.

Now we claim that z is a vertex and $z \in Q_\alpha$. We prove that by the way of contradiction, Assume that z is an edge, then z has a successor incident vertex s in P_α. Hence, z is the only point which larger than (or equal) to itself but not to s in P_α. When $z \notin (P_\alpha \cap Q_\alpha)$, implies that $s \in (P_\alpha \cap Q_\alpha)$ and s is a lower bound larger than z , that is contradiction. Therefore, $z \in Q_\alpha$ and has two incident vertices in it. But if they were in P_α, then one of them is a processor of z which implies a contradiction. Therefore, one of them be in Q_α, and z has two incident vertices in P_α. That means z has three incident vertices which contradict with X is an α-topologized graph.

To prove that z must belong to Q_α, assume that $z \notin Q_\alpha$. But Q_α is α-closed by (3), since X is an α-topologized graph. So, z can not be an α-limit point of Q_α , i. e there exists an α-neighbourhood $N \neq Q_\alpha$ of z. Hence $N \cap P_\alpha$ is an α-neighbourhood of z in P_α , and contains a set of the form $\langle m, M\rangle$ such that $m \leq_P z \leq_P M$. Now for any $x \in P_\alpha$, if $x \leq_P M$, then either $x \leq_P z$,when $x \notin Q_\alpha$ by definition of z or $x \in [z, M)$, when $x \notin Q_\alpha$ since Q_α, and (m, M) are disjoint. So, for any $x \in (P_\alpha \cap Q_\alpha)$, if $z \leq_P M \leq_P x$, then it contradicts with definition of z.

Let $R = (P_\alpha \cup Q_\alpha) = [a,z]_{P_\alpha} \cup [z,c]_{Q_\alpha}$ is α-connected. Take $x \in R \setminus \{a, c\}$, and suppose that $x \in (a,z]_{P_\alpha}$.

Now in $R \setminus \{x\}$, we claim that $[a,x)_{P_\alpha}$ is α-open and prove it by contradiction. If $[a,x)_{P_\alpha}$ is not α-open, then there exist some points p in $[a,x)_{P_\alpha}$ such that the intersection of all neighbourhoods of p with $(R \setminus \{x\}) \setminus [a,x)_{P_\alpha} \neq \emptyset$ (i. e $(x,z]_{P_\alpha} \cup [z,c]_{Q_\alpha} \neq \emptyset$). Since $[a,x)_{P_\alpha}$ is α-open in $[a,z]_{P_\alpha} \setminus \{p\}$, and every neighbourhood of p is different from $(x,z]_{P_\alpha}$,therefore all neighbourhoods of p contains points in $[z,c]_{Q_\alpha}$. But $p \notin [z,c]_{Q_\alpha} = Q_\alpha$, since $p \leq_P x \leq_P z$. So p is an α-limit point of $[z,c]_{Q_\alpha}$, but z is a vertex implies that $[z,c]_{Q_\alpha}$ is an α-path in X. Hence, $[z,c]_{Q_\alpha}$ is α-closed by (7), that is contradiction.

Also, in $R \setminus \{x\}$, we claim that $(x,z]_{P_\alpha} \cup [z,c]_{Q_\alpha}$ is α-open and prove it by contradiction. If it is not α-open, then then there exist some points q in $(x,z]_{P_\alpha} \cup [z,c]_{Q_\alpha}$ such that any neighbourhood of q contains some points in $[a,x)_{P_\alpha}$. So, for $q \in (x,z]_{P_\alpha}$ that is not possible Since $(x,z]_{P_\alpha}$ is α-open in $[a,z]_{P_\alpha}$. Hence, $q \in (z,c]_{Q_\alpha}$that leads to be q an α-limit point of $[a,x)_{P_\alpha} \subseteq [a,z]_{P_\alpha}$. But $q \notin [a,z]_{P_\alpha}$, that is contradiction.

Finally, we obtain an α-separation of $R \setminus \{x\} = \{[a,x)_{P_\alpha}, (x,z]_{P_\alpha} \cup [z,c]_{Q_\alpha}\}$. Since $[a,z]_{P_\alpha}, [z,c]_{Q_\alpha}$ are compact α-paths, then R is compact α-prepath and certainly is an α-path. Hence, $P_\alpha \cup Q_\alpha$ contains an ac- α-path.

The next definition introduces the weaker property than α-Hausdorff property.

Definition 12. Let X be an α-topological space, then X is said to be α-weakly Hausdorff, if for any disjoint points x, y, there exist two α-neighbourhoods U_x, U_y such that $x \in U_x$, $y \in U_y$, then $|U_x \cap U_y|$ is finite. If the intersection is exactly in one point, X is α-almost Hausdorff topological spaces.

Corollary 3.10. If X is any α-topological space, then:

1. Every α-Hausdorff space is α-almost Hausdorff space.
2. Every α-almost Hausdorff space is α-weakly Hausdorff space.
3. Every α-weakly Hausdorff space is α-feebly Hausdorff space.

Proof: this proof is directly from the definitions of these concepts.

The converse of the above corollary is not true in general. In the next counter example, we show that α-feebly Hausdorff space is not α-Hausdorff space.

Example 1. Let $X = V_G \cup E_G = \{x, y\}$ be an an α-topological space with α-topology of X, $\tau_{\alpha X} = \{\emptyset, X, \{x\}\}$ which all elements are α-open sets and their complements are α-closed set. X is α-feebly Hausdorff space, since for any x, y there exist α-open sets $U_x = \{x\}$ and $U_y = X$ of x and y such that $U_x \cap U_y = \{x\} \cap X \subseteq x^{\alpha\diamond} = \{x\}$. Hence, we can not find α-open set containing y but not containing x, therefore X is not α-Hausdorff space.

The following theorem gives us the necessary conditions to obtain the reverse of corollary (10).

Theorem 4. If X is α-feebly Hausdorff topological space, then:

1. X is α-weakly Hausdorff if and only if X is finitely adjacent.
2. X is α-almost Hausdorff if and only if X is uniquely adjacent.
3. X is α-Hausdorff if and only if X is α-T_1.

Proof:

1. Suppose that X is α-weakly Hausdorff, then for any two distinct points x, y, there exist two α-open sets U_x, U_y such that $|U_x \cap U_y|$ is finite. So, $|U_x \cap U_y| \subseteq U_x \cap U_y \subseteq x^{\alpha\diamond} \cap y^{\alpha\diamond} \subseteq x^{\alpha\diamond}$, since X is α-feebly Hausdorff. Hence, $x^{\alpha\diamond} \cap y^{\alpha\diamond}$ is finite which implies that X is finitely adjacent.

Conversely, assume that X is finitely adjacent, then $x^{\alpha\diamond} \cap y^{\alpha\diamond}$ is finite for all distinct points x and y. Since, X is α-feebly Hausdorff, then for any two distinct points x, y, there exist two α-open sets U_x, U_y such that $U_x \cap U_y \subseteq x^{\alpha\diamond}$. But, $x^{\alpha\diamond} \cap y^{\alpha\diamond} \subseteq x^{\alpha\diamond}$, and it is finite, then $|U_x \cap U_y|$ is finite. So, X is α-weakly Hausdorff.

2. Assume that X is α-almost Hausdorff, then for any two distinct points x, y, there exist two α-open sets U_x, U_y such that U_x intersected with U_y in at most one point. Since X is α-feebly Hausdorff, then $U_x \cap U_y \subseteq x^{\alpha\diamond}$, and $x^{\alpha\diamond} \cap y^{\alpha\diamond} \subseteq x^{\alpha\diamond}$. Hence, $x^{\alpha\diamond} \cap y^{\alpha\diamond}$ contains at most one point which implies that X is uniquely adjacent.

Conversely, suppose that X is uniquely adjacent, then $x^{\alpha\diamond} \cap y^{\alpha\diamond}$ consists of at most one point for all distinct points x and y. Since, X is α-feebly Hausdorff, then for any two distinct points x, y, there exist two α-open sets U_x, U_y such that $U_x \cap U_y \subseteq x^{\alpha\diamond}$. But, $x^{\alpha\diamond} \cap y^{\alpha\diamond} \subseteq x^{\alpha\diamond}$, implies that U_x intersected with U_y in at most one point. Hence X is α-almost Hausdorff.

3. Suppose X is α-Hausdorff, then for any two distinct points x, y, there exist two disjoint α-open sets U_x, U_y such that $U_x \cap U_y = \emptyset$. So, for any distinct point x, y, there exist two disjoint α-open sets U_x, U_y in X with $x \in U_x$, $y \notin U_x$ and $y \in U_y$, $x \notin U_y$which means that X is α-T_1.

Conversely, assume that X is α-T_1, then for any distinct point x, y, there exist two disjoint α-open sets U_x, U_y in X with $x \in U_x$, $y \notin U_x$ and $y \in U_y$, $x \notin U_y$. Since, X is α-feebly Hausdorff, then $U_x \cap U_y \subseteq x^{\alpha\diamond}$ for any distinct points x and y. Since, every singleton is α-closed in α-T_1 space, then $\alpha\text{-}\rho(x) = \emptyset$, therefore $x^{\alpha\diamond} = x$. But $x \in U_x$ and $x \notin U_y$, so $U_x \cap U_y = \emptyset$. Hence, X is α-Hausdorff.

The relationship between α-prepath and α-almost Hausdorff is shown in the next theorem.

Theorem 5. An α-prepath is α-almost Hausdorff.

Proof: Assume that x and y are two distinct points in an α-prepath P_α, with a specified α-compatible total order. Now we have two case, either one of these points is an edge, say y, then {y} and P_α are two α-neighbourhoods intersected by exactly one point. Or else, both points are vertices, then there exists an edge or vertex in between, say z. In the last case, the α-open sets $(-\infty, z\rangle$ and $\langle z, \infty)$ intersect in at most one point, and these are the necessary α-neighbourhoods that we wanted. Hence, P_α is α-almost Hausdorff.

The next simple corollary comes directly from the relationship between the concepts of α-almost Hausdorff, α-weakly Hausdorff and α-feebly Hausdorff.

Corollary 3.14. Let X be an topological space, then:

1. An α-prepath is α-weakly Hausdorff.
2. An α-prepath is α-feebly Hausdorff.

Proof : It is trivial.

CONCLUSION

In this paper, alpha feebly Hausdorff spaces in alpha topological spaces are defined, also some properties and relationships are studied in order to continuum the future works to study new related concepts like alpha feebly regular Hausdorff spaces and alpha feebly normal Hausdorff spaces.

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