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Research Article

Dynamical Behaviours of Stage-Structured Fractional-Order Prey-Predator Model with Crowley–Martin Functional Response

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In this paper, the dynamic behaviour of the stage-structure prey-predator fractional-order derivative system is considered and discussed. In this model, the Crowley–Martin functional response describes the interaction between mature preys with a predator. The existence, uniqueness, non-negativity, and the boundedness of solutions are proved. All possible equilibrium points of this system are investigated. The sufficient conditions of local stability of equilibrium points for the considered system are determined. Finally, numerical simulation results are carried out to confirm the theoretical results.

1. Introduction

The subject of fractional integral and derivative is as old as classical calculus. This type of calculus is more comprehensive than the differ-integral calculus due to its worldwide applications in biology as well as in different branches of science, engineering, several ecological models, and some other interdisciplinary fields [1–4]. Many researchers have performed and investigated models of fractional derivatives [5, 6]. Some types of fractional derivatives like Caputo, Caputo–Fabrizio, Riemann–Liouville, and Marchand are powerful mathematical tools for modelling biological systems which cannot be designed by integer derivatives because fractional-order derivatives are not only dependent on the initial conditions but also on the memory of the system [7, 8]. The Lotka–Volterra equations are the first equations that described the dynamics of biological systems, which are also called the predator-prey equations. A stage-structured predator-prey model is considered and elucidated by the authors in Reference [9]. Some authors [10–16] have studied two species models with stages structured in the predators. They discussed and investigated the stability of the all equilibrium points as well as the dynamic behaviour of their systems. In recent years, the global and local dynamics behaviour of a stage-structured predator-prey model is investigated, and the optimal control of harvesting is discussed in References [17–20]. The predator-prey functional response is very important to determine the relationship between the predator and prey for that there are many types of functional responses, namely, Holling types I, II, III, and IV; Michaelis–Menten ratio-dependent type; and Beddington–DeAngelis type of them have been considered and analysed in References [21–27]. The Crowley–Martin functional response is simplified to Michaelis–Menten which has involved one more term explaining mutual interferences of predators for the case where the predator feeding rate is decreased by a higher predator density even when the prey density is high [28, 29]. The subject of the optimal harvesting is very important in managing the renewable resources due to the economic aspect and to keep population at level far from the extinct, so that many researchers and authors have widely investigated and studied this subject in their works, see [30–34] and the references therein.

In this work, we have considered and developed a fractional-order with the Caputo fractional derivative model for a stage-structured prey-predator model with Crowley–Martin functional response and linear harvesting for mature prey species only. The current paper is divided as followsSection 2 contains the main definitions of fractional derivatives and theories for the local stability of the equilibrium points that are used throughout this work. In Section 3, the model is formulated with the existence of all its equilibria. The existence and boundedness of its solution are proved and shown. In Section 4, the conditions for the local stability of all equilibrium points are established. Section 5 presents and confirms the numerical simulation for the theoretical results, while the conclusions and discussions are given and elucidated in Section 6.

2. Main Concepts

Definition 1 (see [35, 36]). The fractional-order derivatives in the meaning of Caputo are defined as follows:

$$D_t^q f(t) = I^{n-q} f^n(t), q > 0, \tag{1}$$

where n is the least integer which is not less than q, I^m is the Riemann–Liouville operator of order m which is given by:

$$I^m = \frac{1}{\Gamma}(m) \int_0^t (t-\tau)^{m-1} d\tau, \tag{2}$$

$m > 0, \Gamma(m)$ is a gamma function.

Some results for the fractional-order derivatives are found in References [21, 25, 37] that are needed throughout this paper.

Lemma 1. *Assume that $f(t) \in C[a,b]$.*

(1) $D_a^q f(t) \in (a,b]$ with $0 < q \le 1$, then, $f(t) = f(a) + 1/\Gamma(q)(D_a^q f)(\zeta)(t-a)^q$.
where $a \le \zeta \le s$ for all $s \in (a,b]$

(2) If $D_t^q f(t) \in C(a,b)$ with $0 < q \le 1$, then we have:

(i) If $D_a^q f(t) \ge 0$, for all $t \in (a,b)$, then $f(t)$ is a nondecreasing function for all $t \in [a,b]$.

(ii) If $D_a^q f(t) \le 0$ for all $t \in (a,b)$ then $f(t)$ is a nonincreasing function for all $t \in [a,b]$.

Lemma 2. *Assume the Cauchy problem.*

$$D_a^q h(t) = \lambda h(t) + f(t), \quad h(a) = b (b \in \mathbb{R}), \tag{3}$$

with $0 < q \le 1$ and $\lambda \in \mathbb{R}$, then the solution of (4) is given by

$$\begin{aligned} h(t) &= bE_q[\lambda(t-a)^q] \\ &+ \int_a^t (t-c)^{q-1} E_{q,q}[\lambda(t-c)^q f(c) dc]. \end{aligned} \tag{4}$$

In the autonomous case:

$$D_a^q h(t) = \lambda h(t), h(a) = b (b \in \mathbb{R}). \tag{5}$$

Then, the solution is $h(t) = bE_q[\lambda(t-a)^q]$ and E_q is the Mittag-Leffler function.

Lemma 3 (see [21]). *Let $w(t)$ be a continuous function on $[t_0, \infty)$ and satisfies*
$D_a^q w(t) \le -\lambda w(t) + \mu$, then the form of the solution of this equation is given by:

$$w(t) \le \left(w_{t_0} - \frac{\mu}{\lambda}\right) E_q[-\lambda(t-t_0)^q] + \frac{\mu}{\lambda}, \tag{6}$$

where $0 < q < 1, (\lambda, \mu) \in \mathbb{R}^2, \lambda \ne 0$, and $t_0 \ge 0$ and t_0 is the initial time.

The next lemma is found in Reference [38] which gives the uniqueness of the solution of fractional-order system.

Lemma 4. *Let $D_t^q u(t) = f(t,u), t > t_0$ be a system fractional-q-order derivatives with the initial condition u_{t_0}, where $0 < q \le 1, f\colon [t_0, \infty) \times \Omega \longrightarrow \mathbb{R}^n$. Then, the above system has a unique solution on $[t_0, \infty) \times \Omega$ if $f(t,u)$ satisfies the locally Lipchitz condition concerning u.*

To examine the stability of the fractional-order system, we need the following.

Definition 2 (see [39]). Let $f(x) = x^n + v_{n-1}x^{n-1} + v_{n-2}x^{n-2} + \ldots\ldots + v_0$ be a polynomial of degree n, then the discriminant of $f(x)$ denoted by $D(f)$ is defined by $D(f) = (-1)^{n(n-1)/2} R(f, f')$, where f' is the derivative of f and $g(x) = x^n + k_{n-1}x^{l-1} + k_{n-2}x^{l-2} + \ldots\ldots + k_0$ and $R(f,g)$ are $(n+l) \otimes (n+l)$ determinants.

Lemma 5 (see [39]). *Consider a characteristic polynomial equation, $f(\lambda) = \lambda^n + v_{n-1}\lambda^{n-1} + v_{n-2}\lambda^{n-2} + \ldots\ldots.. + v_0$, then:*

(1) If $n = 2$, then the conditions for $|arg\lambda_i| > q\pi/2$, for $i = 1,2$ $q \in (0,1)$ are either Routh–Hurwitz conditions or $v_1 < 0, 4v_0 > v_1^2$ and

$$\tan^{-1}\left|\sqrt{\frac{(4v_0 - v_1^2)}{v_1}}\right| > \frac{q\pi}{2}. \tag{7}$$

(2) If $n = 3$, then we have the following cases:

(i) $D(f) > 0 v_2 > 0, v_0 > 0 v_1 v_2 > v_0$, then $|arg\lambda_i| > q\pi/2$, $i+1,2,3$ for $q \in (0,1)$, where $D(f) = 18v_2v_1v_0 + (v_2v_1)^2 - 4v_0(v_2)^3 - 4(v_1)^3 - 27(v_0)^2$

(ii) If $D(f) < 0, v_2 \ge 0, v_1 \ge 0, v_0 > 0$, then $|arg\lambda_i| > q\pi/2 i = 1, 2, 3$ for $q < 2/3$.

3. The Fractional-Order Model

In Reference [40], the author considered the following model:

$$\left.\begin{array}{l}\dfrac{\mathrm{d}}{\mathrm{d}T}\left(X_1(T)\right)=RX_2(T)-D_1X_1(T)-BX_1(T)\\[2ex]\dfrac{\mathrm{d}}{\mathrm{d}T}\left(X_2(T)\right)=BX_1(T)-D_2X_2(T)-B_1X_2^2(T)-\dfrac{A_1X_2(T)Y_2(T)}{\left(1+\alpha X_2(T)\right)\left(1+\beta Y_2(T)\right)}\\[2ex]\dfrac{\mathrm{d}}{\mathrm{d}T}\left(Y_1(T)\right)=\dfrac{A_2X_2(T)Y_2(T)}{\left(1+\alpha X_2(T)\right)\left(1+\beta Y_2(T)\right)}-D_3Y_1(T)-H_1Y_1(T)\\[2ex]\dfrac{\mathrm{d}}{\mathrm{d}T}\left(Y_2(T)\right)=H_1Y_1(T)-D_4Y_1(T)-B_2Y_2^2(T).\end{array}\right]. \tag{8}$$

$X_1(T), X_2(T)$ are the densities of immature and mature prey species, respectively. $Y_1(T)$and$Y_2(T)$ represent the densities of immature and mature predator species respectively. The parameter R is the birth rate of immature prey; B and H_1 indicate maturity rate of immature prey and immature predator, respectively; B_1 and B_2 express the competition rate between a mature prey population and mature predator population, respectively; D_1, D_2, D_3, D_4 are the death rates of immature and mature prey, the death rates of immature and mature predators, respectively. A_1 is the maximum value which per capita reduction rate of the mature prey can attain, while A_2 is the predator conversation rate. α and β are measures of the half-saturation of prey species and the coefficient of interference among predators at a high density of mature prey, respectively. All that parameters are positive.

In this work, the system (8) is modified and then fractional order derivative is introduced in the modified system with the Caputo-type derivative. We also consider the harvesting in the mature prey only. The modified system is given as follows:

$$\left.\begin{array}{l}\dfrac{\mathrm{d}}{dT}\left(X_1(T)\right)=RX_2(T)-D_1X_1(T)-BX_1(T)\\[2ex]\dfrac{\mathrm{d}}{\mathrm{d}T}\left(X_2(T)\right)=BX_1(T)-D_2X_2(T)-\dfrac{A_1X_2(T)X_3(T)}{\left(1+\alpha X_2(T)\right)\left(1+\beta X_3(T)\right)}-HX_2(T)\\[2ex]\dfrac{d}{dT}\left(X_3(T)\right)=\dfrac{A_2X_2(T)X_3(T)}{\left(1+\alpha X_2(T)\right)\left(1+\beta X_3(T)\right)}-D_3X_3(T)\end{array}\right]. \tag{9}$$

The parameter H denotes the harvesting rate of mature prey species. Next, we reduce the parameters of the modified system (9) by using the following nondimensional transformation:

$$x_1=\frac{BX_1}{RD_2}, x_2=\alpha X_2, x_3=\beta X_3, t=D_1T. \tag{10}$$

Therefore, we obtain a new system which has the following form:

$$\left.\begin{array}{l}\dfrac{\mathrm{d}}{\mathrm{d}t}\left(x_1(t)\right)=rx_2(t)-(1+b)x_1(t)\\[2ex]\dfrac{\mathrm{d}}{\mathrm{d}t}\left(x_2(t)\right)=d_1x_1(t)-d_2x_2(t)-\dfrac{a_1x_2(t)x_3(t)}{\left(1+x_2(t)\right)\left(1+x_3(t)\right)}\\[2ex]\dfrac{\mathrm{d}}{\mathrm{d}t}\left(x_3(t)\right)=\dfrac{a_2x_2(t)x_3(t)}{\left(1+x_2(t)\right)\left(1+x_3(t)\right)}-d_3x_3(t)\end{array}\right], \tag{11}$$

where

$$r = \frac{B}{\alpha D_2 D_1}, b = \frac{B}{D_1}, d_1 = \frac{RD_2\alpha}{D_1}, d_2 = \frac{D_2}{D_1}, a_1 = \frac{A_1}{D_1\beta},$$
$$d_3 = \frac{D_3}{D_1}, a_2 = \frac{A_2}{D_1\alpha}. \tag{12}$$

Now, we present the fractional-order derivatives q in model (3) with the Caputo-type derivatives and then the system (11) will be as follows:

$$\left.\begin{aligned} D_t^q(x_1(t)) &= rx_2(t) - (1+b)x_1(t) \\ D_t^q(x_2(t)) &= d_1x_1(t) - d_2x_2(t) - \frac{a_1x_2(t)x_3(t)}{(1+x_2(t))(1+x_3(t))} - hx_2(t) \\ D_t^q(x_3(t)) &= \frac{a_2x_2(t)x_3(t)}{(1+x_2(t))(1+x_3(t))} - d_3x_3(t) \end{aligned}\right]. \tag{13}$$

With $x_1(t) \geq 0, x_2(t) \geq 0, x_3(t) \geq 0$.

The following theorem gives the boundedness and nonnegativity of the solution of the system (13).

Theorem 1. *For the fractional-order system (13), we have the following:*

(1) All solutions that start in $F_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$ are non-negative solutions.

(2) If $1 + b > rd_1/d_2$, then all solutions that start in F_+ are bounded.

Proof.

(1) To prove the solution of the fractional-order system (13) is nonnegative, we start with $x_1(0) > 0 \text{for} t = 0$ assuming that $x_1(t) \geq 0, \text{for} t \geq 0$ is not true; therefore, there exists $t_* > 0$ such that

$$\left.\begin{aligned} &x_1(t) > 0, 0 \leq t < t_* \\ &x_1(t) = 0, t = t_* \\ &x_1(t) < 0, t > t_* \end{aligned}\right\}. \tag{14}$$

So, from the first equation of the system (13), we obtain the following:

$$D_t^q x_1(t)|_{t=t_*} = 0. \tag{15}$$

According to part (1) of Lemma 1, we get $x_1(t_*^+) = 0$ and this is contradiction because $x_1(t_*^+) < 0$, that is $x_1(t) < 0, t > t_*$.

Therefore, $x_1(t) \geq 0$, for all$t \geq 0$. In the same manner, we can prove that $x_2(t) \geq 0, x_3(t) \geq 0$, for all$t \geq 0$.

(2) Let $F_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$, and x_1, x_2, and x_3 in $\mathbb{R}^3$, define a function $V_t = x_1(t) + r/d_2x_2(t) + ra_1/d_2a_2ra_1/d_2a_2x_3(t)$, it follows that

$$\begin{aligned} D^qV(t) &= D^qx_1(t) + \frac{r}{d_2}D^qx_2(t) + \frac{ra_1}{d_2a_2}D^qx_3(t) \\ &= rx_2 - (1+b)x_1 \\ &\quad + \frac{r}{d_2}\left(d_1x_1 - (d_2+h)x_2 - \frac{a_1x_2x_3}{(1+x_2)(1+x_3)}\right) \\ &\quad + \frac{ra_1}{d_2a_2}\left(\frac{a_2x_2x_3}{(1+x_2)(1+x_3)} - d_3x_3\right), \\ &= \left(\frac{rd_1}{d_2} - (1+b)\right)x_1 - \frac{r}{d_2}(h)x_2 - \frac{ra_1}{d_2a_2}(d_3)x_3. \end{aligned} \tag{16}$$

Now, for each $k > 0$, we have

$$\begin{aligned} D^qV(t) + kV(t) &= \left(\frac{\mathrm{rd}_1}{\mathrm{d}_2} - (1+b)\right)x_1 - \frac{r}{\mathrm{d}_2}(h)x_2 \\ &\quad - \frac{ra_1}{\mathrm{d}_2a_2}(d_3)x_3 + kx_1 + k\frac{r}{\mathrm{d}_2}x_2 + k\frac{ra_1}{\mathrm{d}_2a_2}x_3 \\ &= \left(k - \left(1 + b - \frac{\mathrm{rd}_1}{\mathrm{d}_2}\right)x_1\right) + \frac{r}{d_2}(k-h)x_2 \\ &\quad + \frac{ra_1}{d_2a_2}(k - d_3)x_3. \end{aligned} \tag{17}$$

If we choose $k < \min\{1 + b - rd_1/d_2, h, d_3\}$, then we get $D^qV(t) + kV(t) \leq \varepsilon$ for some $\varepsilon > 0$.

Then, from Lemma 3, we obtain that $V(t) \leq (V(0) - (\varepsilon/k))E_q[-k(t-0)^q] + v/K$. Therefore, we have $0 \leq V(t) \leq \varepsilon/k$ as $t \longrightarrow \infty$ and all solutions of system (13) are bounded.

The next theorem gives the existence and uniqueness of the solution system (13). □

Theorem 2. *The existence and uniqueness solution of system (4) are given for each non-negative initial condition:*

Proof. Let the region be defined by $F\times(0,\mathrm{T}]$, $T<\infty$.

Where $F=(x_1,x_2,x_3)\in\mathbb{R}_+^3$: $\max(|x_1|,|x_2|,|x_3|)\le M$.

We assume that $X=(x_1,x_2,x_3),\dot{X}=(\dot{x_1},\dot{x_2},\dot{x_3})$ and $B(X)=(B_1(X),B_2(X),B_3(X))$ be a mapping, such that

$$\begin{aligned} B_1(X)&=rx_2-(1+b)x_1,\\ B_2(X)&=d_1x_1-(d_2+h)x_2-\frac{a_1x_2x_3}{(1+x_2)(1+x_3)},\\ B_3(X)&=\frac{a_2x_2x_3}{(1+x_2)(1+x_3)}-d_3x_3. \end{aligned} \tag{18}$$

For $X,\dot{X}\in F$, one can get the following:

$$\begin{aligned} B(X)-B(\dot{X})=&\left|B_1(X)-B_1(\dot{X})\right|+\left|B_2(X)-B_2(\dot{X})\right|+\left|B_3(X)-B_3(\dot{X})\right|\\ =&\left|r(x_2-\dot{x_2})-(1+b)(x_1-\dot{x_1})\right|\\ &+\left|d_1(x_1-\dot{x_1})-(d_2+h)(x_2-\dot{x_2})-a_1\left(\frac{x_2x_3(1+\dot{x_2})(1+\dot{x_3})-\dot{x_2}\dot{x_3}(1+x_2)(1+x_3)}{(1+x_2)(1+x_3)(1+\dot{x_2})(1+\dot{x_3})}\right)\right|\\ &+\left|-d_3(x_3-\dot{x_3})+a_2\left(\frac{x_2x_3(1+\dot{x_2})(1+\dot{x_3})-\dot{x_2}\dot{x_3}(1+x_2)(1+x_3)}{(1+x_2)(1+x_3)(1+\dot{x_2})(1+\dot{x_3})}\right)\right|. \end{aligned} \tag{19}$$

Since $1/(1+x_2)(1+x_3)(1+\dot{x_2})(1+\ \dot{x_3})\le 1,\max(|x_1|,|x_2|,|x_3|)\le M$, we have.

Where $L=\max\{1+b+d_1,(r+d_2+h+a_1+a_2)(M(1+M)),\ (d_3+(a_1+a_2)M(1+M))\}$. Therefore, $B(X)$ satisfies the Lipchitz condition, so that the system (4) has a unique solution. □

4. Local Stability Analysis

In this section, we investigate the existence of the equilibrium points of the system (4). We also give the conditions for the local stability of its equilibria.

Theorem 3 (see [41, 42]). *Consider the following fractional-order differential system:*

$$D_t^q(\overrightarrow{x(t)})=\overrightarrow{f}(\overrightarrow{x}). \tag{20}$$

$\overrightarrow{x(0)}=\overrightarrow{x_0}$, *$q\in(0,1)$, and $\overrightarrow{x}\in\mathbb{R}^n$. The point $\overrightarrow{x^*}$ that satisfies $\overrightarrow{f}(\overrightarrow{x^*})=0$ is called the equilibrium point of the system (20). It is called locally asymptotically stable if $|arg(\lambda_i)|>q\pi/2$ for $i=1,2,\ldots..n$, where λ_i are the eigenvalues of the Jacobian matrix which are evaluated at $\overrightarrow{x^*}$. Otherwise, it is called an unstable point.*

To find all possible equilibrium points of the system (4), we have to solve the following equations:

$$\begin{aligned} D_t^qx_1(t)&=0,\\ D_t^qx_2(t)&=0,\\ D_t^qx_3(t)&=0. \end{aligned} \tag{21}$$

Therefore, the system (4) has three possible equilibrium points, namely:

(1) The trivial equilibrium point $e_0=(0,0,0)$ always exists.

(2) The free predator point $e_1=(x_1^*,x_2^*,0)$ exists if

$$x_2^*=\left(\frac{1+b}{r}\right)x_1^*,r=\frac{d_2+h+d_2b+hb}{d_1}. \tag{22}$$

(3) The interior equilibrium pointe$_2=(x_3^*,x_3^*,x_3^*)$ exists only if

$$\frac{\mathrm{d}1r}{1+b}>d_2+h,\frac{a_1x_3^*}{(1+x_3^*)(d1r/1+b-d_2-h)}>1. \tag{23}$$

where $x_2^*=a_1x_3^*/(1+x_3^*)((d1r/1+b)-d_2-h)-1$, $x_1^*=((r/1+b)x_2^*,x_3^*$ is the positive root of the following equation:

$$x_2^*+ax_2^*+b_1=0. \tag{24}$$

Here, $a=a_1d_3-a_1a_2+a_2(d1r/1+b–d_2-h)/a_1d_3$ and $b_1=a_2(d1r/1+b-d_2-h)/a_1d_3$.

Since $b_1 > 0$ if $a < 0$ then, equation (24) may have two positive real roots.

The Jacobian matrix of fractional-order system (4) associated with arbitrary fixed point (x_1, x_2, x_3) is given by

$$J(x_1, x_2, x_3) = \begin{bmatrix} -(1+b) & r & 0 \\ d_1 & -(d_2+h) - \dfrac{a_1 x_3}{(1+x_3)(1+x_2)^2} & \dfrac{-a_1 x_2}{(1+x_3)(1+x_2)^2} \\ 0 & \dfrac{a_2 x_3}{(1+x_3)(1+x_2)^2} & \dfrac{a_2 x_2}{(1+x_2)(1+x_3)^2} - d_3 \end{bmatrix}. \tag{25}$$

So, the general characteristic equation of (25) is as follows:

$$f(\lambda) = \lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 = 0. \tag{26}$$

where

$$p_2 = \left(1 + b + d_2 + h + \frac{a_1 x_3}{(1+x_3)(1+x_2)^2}\right) - \left(\frac{a_2 x_2}{(1+x_2)(1+x_3)^2} - d_3\right), \tag{27}$$

$$\begin{aligned} p_1 &= \frac{a_1 a_2 x_3 x_2}{(1+x_2)^3(1+x_3)^3} - r d_1 - \left(d_2 + h + \frac{a_1 x_3}{(1+x_3)(1+x_2)^2}\right)\left(\frac{a_2 x_2}{(1+x_2)(1+x_3)^2} - d_3\right) \\ &\quad - (1+b)\left(\left(d_2 + h + \frac{a_1 x_3}{(1+x_3)(1+x_2)^2}\right) + \left(\frac{a_2 x_2}{(1+x_2)(1+x_3)^2} - d_3\right)\right), \\ p_0 &= (1+b)\left(\frac{a_1 x_2}{(1+x_2)(1+x_3)^2}\right)\left(\frac{a_2 x_3}{(1+x_3)(1+x_2)^2}\right) + r d_1\left(\frac{a_2 x_2}{(1+x_2)(1+x_3)^2} - d_3\right) \\ &\quad - (1+b)\left(d_2 + h + \frac{a_1 x_3}{(1+x_3)(1+x_2)^2}\right)\left(\frac{a_2 x_2}{(1+x_2)(1+x_3)^2} - d_3\right). \end{aligned} \tag{28}$$

To analyse the local stability of the fixed points e_0, e_1, e_2, we give the following theorem.

Theorem 4. *For the system (4), we have the following:*

(1) *If $d_2 + h + bd_2 + bh > rd_1$, then the point e_0 is a locally stable point.*

(2) *The free predator point e_1 is always an unstable point.*

(3) *The interior equilibrium point e_2 is locally asymptotically stable if one of the following conditions hold.*

  (i) *If $D(f) > 0$, $p_2 > 0, p_0 > 0$ and $p_1 p_2 > p_0$ for $q \in (0, 1)$.*

  (ii) *If $D(f) < 0, p_2 \geq 0, p_1 \geq 0, p_0 > 0$ for $q < 2/3$, where p_2, p_1, and p_0 are defined in equation(29).*

Proof. (1) For the point e_0, the Jacobian matrix $J(e_0)$ is given as follows:

$$J(e_0) = \begin{bmatrix} -(1+b) & r & 0 \\ d_1 & -(d_2+h) & 0 \\ 0 & 0 & -d_3 \end{bmatrix}. \tag{29}$$

So, the characteristic equation is $P(\lambda) = (-d_3 - \lambda)(\lambda^2 + u_1\lambda + u_0)$.

Where $u_1 = (1 + b + d_2 + h)$ and $u_0 = (d_2 + h + bd_2 + bh - rd_1)$. The roots of the above characteristic equation are $\lambda_1 = (-d_3)$. It is clear that $|\arg\lambda_1| = \pi > q\pi/2$.

Since $u_1 > 0$ and if $d_2 + h + bd_2 + bh > rd_1$ that means $u_0 > 0$, then the Routh–Hurwitz conditions are satisfied.

According to part 1 of Lemma 5, e_0 is stable.

(2) It is clear that the Jacobian matrix $J(e_1)$ is as follows:

$$J(e_1) = \begin{bmatrix} -(1+b), & \dfrac{d_2+h+d_2b+hb}{d_1}, & 0, \\ d_1, & -(d_2+h), & \dfrac{-a_1(1+b/r)x_1^*}{(1+(1+b/r)x_1^*)}, \\ 0, & 0, & \dfrac{a_2(1+b/r)x_1^*}{(1+(1+b/r)x_1^*)} - d_3. \end{bmatrix}. \tag{30}$$

And the corresponding characteristic equation is as follows:

$$P(\lambda) = \left(\frac{a_2((1+b/r))x_1^*}{(1+((1+b/r))x_1^*)} - d_3 - \lambda\right) \cdot \lambda(\lambda + 1 + b + d_2 + h) = 0. \tag{31}$$

Therefore, one of the eigenvalues is equal to zero. Thus, the free predator point e_1 is unstable.

(3) The Jacobian matrix at the interior equilibrium point e_2 is given as follows:

$$J(e_2) = \begin{bmatrix} -(1+b) & r & 0 \\ d_1 & -(d_2+h) - \dfrac{a_1x_3^*}{(1+x_2^*)(1+x_3^*)^2} & \dfrac{-a_1x_2^*}{(1+x_2^*)(1+x_3^*)^2} \\ 0 & \dfrac{a_2x_3^*}{(1+x_2^*)(1+x_3^*)^2} & \dfrac{a_2x_2^*}{(1+x_2^*)(1+x_3^*)^2} - d_3 \end{bmatrix}. \tag{32}$$

So, the characteristic polynomial of $J(e_2)$ can be obtained:

$$f(\lambda) = \lambda^3 + p_2\lambda^2 + p_1\lambda + p_0, \tag{33}$$

where

$$\begin{aligned} p_2 &= \left(1 + b + d_2 + h + \frac{a_1x_3^*}{(1+x_3^*)(1+x_2^*)^2}\right) + \left(d_3 - \frac{a_2x_3^*}{(1+x_2^*)(1+x_3^*)^2}\right), \\ p_1 &= \frac{a_1a_2x_3^*x_2^*}{(1+x_3^*)^3(1+x_3^*)^3} - rd_1 - \left(d_2 + h + \frac{a_1x_3^*}{(1+x_3^*)(1+x_2^*)^2}\right)\left(\frac{a_2x_2^*}{(1+x_2^*)(1+x_3^*)^2} - d_3\right) \\ &\quad + (1+b)\left(d_2 + h + \frac{a_1x_3^*}{(1+x_3^*)(1+x_2^*)^2} + \left(d_3 - \frac{a_2x_2^*}{(1+x_2^*)(1+x_3^*)^2}\right)\right), \\ p_0 &= (1+b)\left(\frac{a_1x_3^*}{(1+x_2^*)(1+x_3^*)^2}\right)\left(\frac{a_2x_3^*}{(1+x_3^*)(1+x_3^*)^2}\right) + rd_1\left(\frac{a_2x_3^*}{(1+x_3^*)(1+x_3^*)^2} - d_3\right) \\ &\quad - (1+b)\left(d_2 + h + \frac{a_1x_3^*}{(1+x_3^*)(1+x_2^*)^2}\right)\left(\frac{a_2x_2^*}{(1+x_2^*)(1+x_3^*)^2} - d_3\right). \end{aligned} \tag{34}$$

Table 1: Parameter values of the equilibrium point e_0.

Parameter	Parameter value
d_1	0.1
b	0.2825
h	0.1
d_3	0.5133
r	6.2416
a_2	0.5851
d_2	0.9872
a_1	0.6621
q	1,0.98, 0.88, 0.78
$x_1(0)$	0.4175
$x_2(0)$	1.9799
$x_3(0)$	0.837

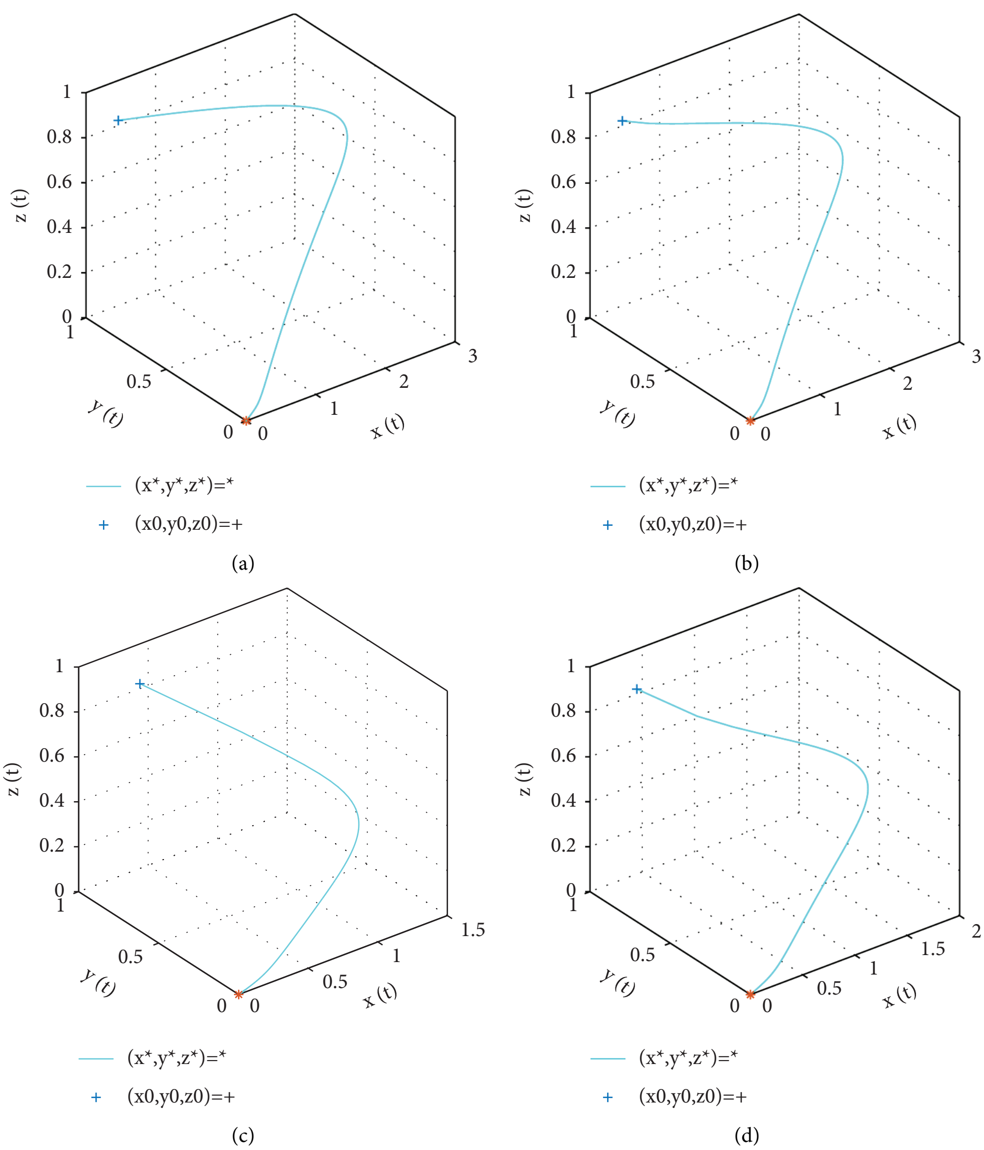

Figure 1: Shows the local stability of the equilibrium point e_0 at which $x(t)$ represents immature prey species, $y(t)$ represents mature prey species, and $z(t)$ represents predator species. The values in Table 1 are used with different values of fractional-order(a) $q = 1$, (b) $q = 0.98$, (c) $q = 0.88$, and (d) $q = 0.78$.

Table 2: Parameter values of the equilibrium point e_2.

Parameter	Parameter value
d_1	0.3299
b	0.2825
h	0.6168
d_3	0.5133
r	6.2416
a_2	0.5851
d_2	0.9872
a_1	0.6621
q	1, 0.98, 0.88, 0.78
$x_1(0)$	41.6
$x_2(0)$	8
$x_3(0)$	0.2

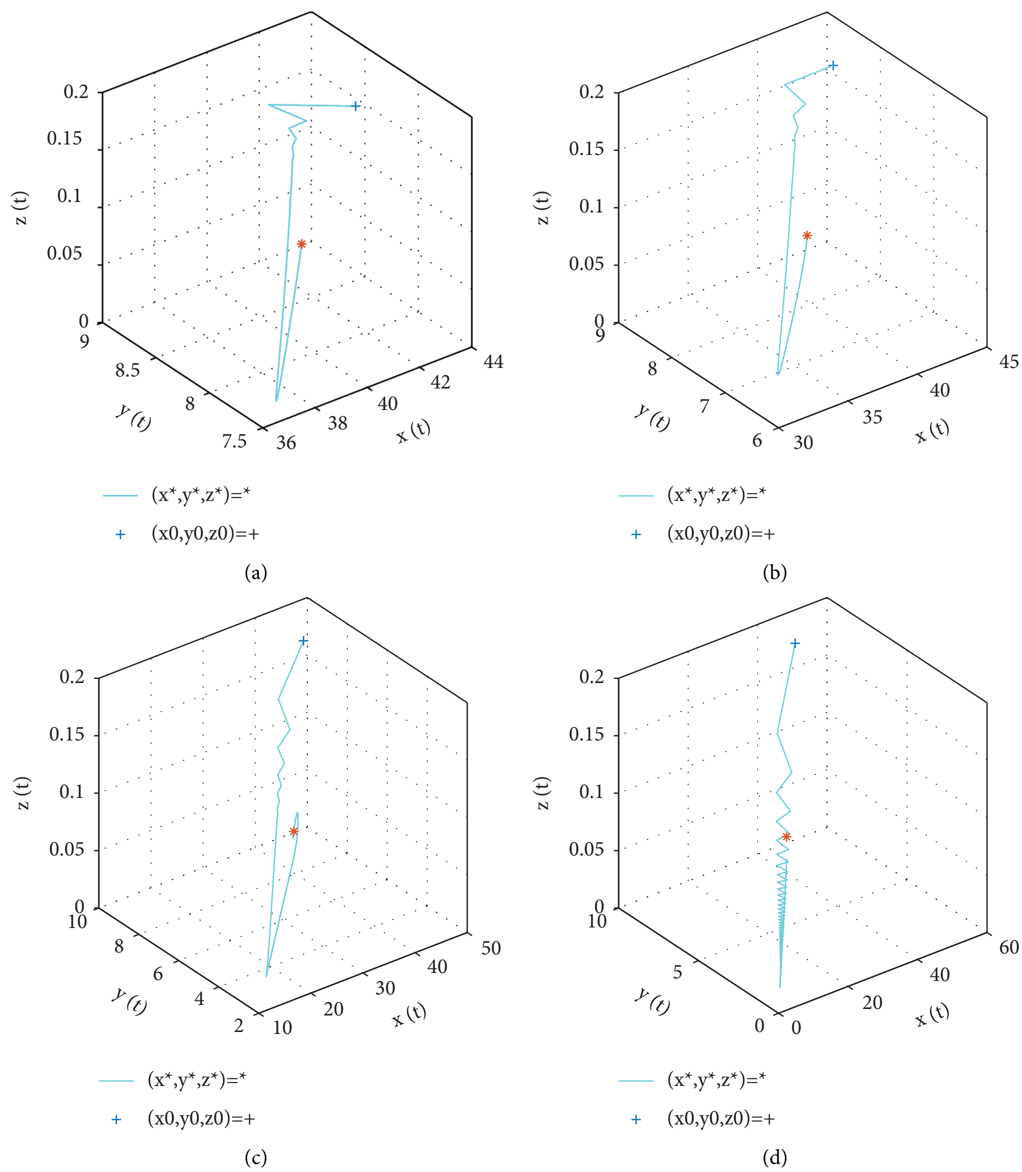

Figure 2: Shows the local stability of the equilibrium point e_3 at which $x(t)$ represents immature prey species, $y(t)$ represents mature prey species, and $z(t)$ represents predator species. The values in Table 2 are used with different values of fractional-order(a) $q = 1$, (b) $q = 0.98$, (c) $q = 0.88$, and (d) $q = 0.78$.

Then, the discriminant $D(f)$ of the cubic polynomial $f(\lambda)$ is the following:

$$
\begin{aligned}
D(f) &= -\begin{vmatrix} 1 & p_2 & p_1 & p_0 & 0 \\ 0 & 1 & p_2 & p_1 & p_0 \\ 3 & 2p_2 & p_1 & 0 & 0 \\ 0 & 3 & 2p_2 & p_1 & 0 \\ 0 & 0 & 3 & 2p_2 & p_1 \end{vmatrix} \\
&= 18p_2p_1p_0 + (p_2p_1)^2 - 4p_0(p_2)^3 - 4(p_1)^3 \\
&\quad - 27(p_0)^2.
\end{aligned} \tag{35}
$$

According to Lemma 5 (2), if $D(f) > 0$, $p_2 > 0$, $p_0 > 0$ and $p_1p_2 > p_0$for q $\in$ (0, 1), or $D(f) < 0$, $p_2 \geq 0$, $p_1 \geq 0$, $p_0 > 0$ for $q < 2/3$.

Then, the interior fixed point e_2 is locally asymptotically stable. □

5. Numerical Simulations

In this section, we give numerical simulations to confirm the theoretical results that are employed in Section 4. For the equilibrium point e_0, the parameter values in Table 1 are considered to carry out that e_0 is locally asymptotically stable. Figure 1 shows that e_0 is locally stable according to conditions in Theorem 4.

For the equilibrium point e_2, the values of parameter are chosen and set in Table 2. According to the conditions in Theorem 4, the point e_2 is locally stable. Figure 2 illustrates the stability of the positive equilibrium point e_2.

6. Conclusions

The fractional-order differential system has been successfully applied in mathematical biology. In this paper, we have discussed the dynamical behaviour of a stage-structure prey-predator with Crowley–Martian functional response and a linear harvesting rate in a fractional-order state. The existences, uniqueness, non-negativity, and the boundedness solutions of the fractional-order system are presented and proved. It is found that the considered system has at least three equilibrium points. One of them are unstable points. For the other points, conditions are set to guarantee their local stability.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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