St-closed Submodule

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Abstract

Throughout this paper R represents commutative ring with identity and M is a unitary left R-module, the purpose of this paper is to study a new concept, (up to our knowledge), named St-closed submodules. It is stronger than the concept of closed submodules, where a submodule N of an R-module M is called St-closed (briefly $N \leq_{Stc} M$) in M, if it has no proper semi-essential extensions in M, i.e if there exists a submodule K of M such that N is a semi-essential submodule of K then N = K. An ideal I of R is called St-closed if I is an St-closed R-submodule. Various properties of St-closed submodules are considered.

Keywords**:** Prime submodules, Essential submodules, Semi-essential submodules, Closed submodules, St-closed submodules, Fully prime modules and fully essential modules.

Introduction

Let R be a commutative ring with identity and let M be a unitary left R-module, and all R-modules under study contains prime submodules. It is well known that a nonzero submodule N of M is called essential (briefly $N \leq_e M$), if $N \cap L \neq (0)$ for each nonzero submodule L of M [8], and a nonzero submodule N of M is called semi-essential (briefly $N \leq_{sem} M$), if $N \cap P \neq (0)$ for each nonzero prime R- submodule P of M [2]. Equivalently, a submodule N of an R-module M is called semi-essential if whenever $N \cap P = (0)$, then P = (0) for every prime submodule P of M [11], where a submodule P of M is called prime, if whenever $rm \in P$ for $r \in R$ and $m \in M$, then either $m \in P$ or $r \in (P :_R M)$ [14].

A submodule N of M is called closed submodule (briefly $N \leq_c M$), if N has no proper essential extensions in M, i.e if $N \leq_e K \leq M$ then N = K [6]. In our work we introduce a new concept (up to our knowledge), named St-closed submodules, which is stronger than the concept of closed submodules, where a submodule N of an R-module M is called St- closed if N has no proper semi-essential extensions in M, i.e if $N \leq_{sem} K \leq M$ then N = K. This paper consist of three sections, in section one we investigate the main properties of St-closed submodules, such as the transitively property. Also we study the relationships between St-closed submodules, closed submodules and y-closed submodules. In S_2 we study the behavior of the class of St-closed submodules in the class of multiplication modules. In S3 we study modules satisfying the chain conditions on St-closed submodules.

S_1: St-closed submodules

In this section we investigate the main properties of St-closed submodules such as the transitive property. Moreover, we study the relationships between St-closed submodules and other submodules.

<u>Definition (1.1):</u>
Let M be an R-module, a submodule N of M is called St-closed in M (briefly $N \leq_{Stc} M$), if N has no proper semi-essential extensions in M, i.e if there exists a submodule K of M such that N is a semi-essential submodule of K then N = K. An ideal I of R is called an St-closed, if it is St-closed R-submodule.

Examples and Remarks (1.2):

1) Consider the Z-module $M = Z_8 \oplus Z_2$. In this module there are eleven submodules which are $<(\bar{0}, \bar{0})>$, $<(\bar{1}, \bar{0})>$, $<(\bar{0}, \bar{1})>$, $<(\bar{1}, \bar{1})>$, $<(\bar{2}, \bar{0})>$, $<(\bar{2}, \bar{1})>$, $<(\bar{4}, \bar{0})>$, $<(\bar{4}, \bar{1})>$, $<(\bar{0}, \bar{1}), (\bar{4}, \bar{0})>$, $<(\bar{2}, \bar{0}), (\bar{4}, \bar{1})>$, and M. The submodules $<(\bar{0}, \bar{1})>$, $<(\bar{4}, \bar{1})>$, and M are St-closed in M, since they have no proper semi-essential extensions in M. On the other hand, the submodules $<(\bar{0}, \bar{0})>$, $<(\bar{1}, \bar{1})>$, $<(\bar{1}, \bar{0})>$, $<(\bar{2}, \bar{0})>$, $<(\bar{2}, \bar{1})>$, $<(\bar{4}, \bar{0})>$, $<(\bar{0}, \bar{1}), (\bar{4}, \bar{0})>$, and $<(\bar{2}, \bar{0}), (\bar{4}, \bar{1})>$, are not St-closed submodules in M, since they have semi-essential extensions in M.

2) Every R-module M is an St-closed submodule in M.

3) (0) may not be St-closed submodule of M, for example $(\bar{0})$ is not St-closed submodule in the Z-module, Z_2.

4) If a submodule N of an R-module M is a semi-essential and an St-closed, then N = M.

5) If N is an St-closed submodule in M then $(N \dot{:}_R M)$ need not be St-closed ideal in R, for example; $(\bar{8})$ is an St-closed submodule in the Z-module Z_{24}, while $((\bar{8}) \dot{:}_Z Z_{24}) = 8Z$ is not St-closed ideal in Z.

6) A direct summand of an R-module M is not necessary St-closed submodule in M, for example: Consider the Z-module, Z_{12}, where $Z_{12} = (\bar{3}) \oplus (\bar{4})$. The direct summand $(\bar{4}) = \{\bar{0}, \bar{4}, \bar{8}\}$ is an St-closed submodule in Z_{12}, since $(\bar{4})$ has no proper semi-essential extensions in Z_{12}. But the direct summand $(\bar{3}) = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\}$ of Z_{12} is not St-closed submodule since $(\bar{3})$ is a semi-essential submodule of Z_{12}. Also the Z-module, $Z_{36} = (\bar{4}) \oplus (\bar{9})$, it is clear that $(\bar{9})$ is a direct summand of Z_{36} but not St-closed submodule in Z_{36}.

7) Let M be an R-module, if $M = A \oplus B$, then even though A or B or both of them are prime submodules of M, then neither A nor B are necessary St-closed submodules in M. For example: the Z-module $Z_{30} = (\bar{5}) \oplus (\bar{6}) = (\bar{2}) \oplus (\overline{15})$, both of $(\bar{2})$ and $(\bar{5})$ are prime submodules of Z_{30} and direct summand, but neither $(\bar{2})$ nor $(\bar{5})$ are St-closed submodules in Z_{30}. In fact both of $(\bar{2})$ and $(\bar{5})$ are semi-essential submodules of Z_{30}.

8) Let M be an R-module, and let A be an St-closed submodule of M. If B is a submodule of M such that $A \cong B$, then it is not necessary that B is an St-closed submodule in M. For example, the Z-module Z is an St-closed submodule in Z, and $Z \cong 3Z$, but 3Z is not St-closed submodule in Z, since 3Z is a semi-essential submodule of Z.

Remarks (1.3):

1) Every St-closed submodule in an R-module M is a closed submodule in M.

<u>Proof</u> (1):

Let N be an St-closed submodule in M, and let $K \leq M$ with $N \leq_e K \leq M$. Since $N \leq_e K$, then $N \leq_{sem} K$ [2, Example (2), P.49]. But N is an St-closed submodule in M, thus N = K, that is N is a closed submodule in M.

The converse is not true in general, for example: In the Z-module Z_{24} we note that $(\bar{3})$ is a closed submodule in Z_{24}, but it is not St-closed. Also $(\bar{9})$ is a closed submodule in Z_{36}, but it is not St-closed in Z_{36}.

2) Let N be an St-closed submodule of M. If B is a relative M-complement of N, then N is a relative M-complement of B, where a relative complement for K in M is any submodule L of M which is maximal with respect to the property $K \cap L = (0)$ [6].

<u>Proposition (1.4):</u>

Let M be an R-module, and let $(0) \neq C \leq M$, then there exists an St-closed submodule H in M such that $C \leq_{sem} H$.

<u>Proof</u>:

Consider the set V = {K| K is a submodule of M such that $C \leq_{sem} K$}. It is clear that $V \neq \emptyset$. By Zorn's Lemma, V has a maximal element say H. In order to prove that H is an St-closed submodule in M; assume that there exists a submodule D of M such that $H \leq_{sem} D \leq M$. Since $C \leq_{sem} H$ and $H \leq_{sem} D$, so by [11, Proposition (1.5)], $C \leq_{sem} D$. But this a contradicts the maximality of H, thus H = D. That is H is an St-closed submodule in M with $C \leq_{sem} H$.

We cannot prove the transitive property for St-closed submodules. However under some conditions we can prove this property as we see in the following result.

Proposition (1.5):

Let A and B be submodules of an R-module C. If A is an St-closed in B and B is an St-closed in C, then A is St-closed in C provided that B contained in (or containing) any semi-essential extension of A.

<u>Proof</u>:

Let $L \leq C$ such that $A \leq_{sem} L \leq C$. By assumption we have two cases: If $L \leq B$, since A is an St-closed submodule in B then A = L, hence A is an St-closed submodule in C. If $B \leq L$, since $A \leq_{sem} L$, so by [2, Proposition 4], $B \leq_{sem} L$. But B is an St-closed in C, thus

B = L. That is A $\leq_{sem}$ B. On the other hand, A is an St-closed submodule in B, so A = B, hence A is an St-closed submodule in C.

Recall that an R-module M is called chained if for each submodules A and B of M either A ≤ B or B ≤ A [13].

Corollary (1.6):

Let M be a chained module, and let A and B be submodules of M such that A ≤ B ≤ M. if A is an St-closed submodule in B and B is an St-closed submodule in M then A is an St-closed submodule in M.

Proof:

Let L ≤ M such that A $\leq_{sem}$ L ≤ M. since M is a chained module, then either L ≤ B or B ≤ L, and the result follows as the same argument which used in the proof of the Proposition (1.5).

We can put other condition to get the transitive property of St-closed submodules, but before that we need to recall some definitions and give some remarks.

Recall that a nonzero R-module M is called fully essential, if every nonzero semi-essential submodule of M is essential submodule of M [12], and an R-module M is called fully prime, if every proper submodule of M is a prime submodule [3], and every fully prime module is a fully essential module [11].

Proposition (1.7):

Let N be a nonzero closed submodule of an R-module M. If every semi-essential extensions of N is a fully essential submodule of M, then N is an St-closed submodule in M.

***Proof*:**

Let N be a nonzero closed submodule of M, and let L ≤ M such that N $\leq_{sem}$ L≤ M. By assumption L is a fully essential module, therefore N $\leq_e$ L. But N $\leq_{co}$ M, thus N = L. That is N $\leq_{Stc}$ M.

Remark (1.8):

If an R-module M is fully prime, then every nonzero closed submodule in M is an St-closed submodule in M.

***Proof*:**

Let N be a nonzero closed submodule of M, and let N $\leq_{sem}$ L ≤ M. Then by [11, Proposition (2.1)], N $\leq_e$ L. But N $\leq_{co}$ M, thus N = L, and we are don.

Proposition (1.9):

Let C be an R-module and let (0) ≠ A ≤ B ≤ C. Assume that every semi-essential extension of A is a fully essential submodule of M. If A $\leq_{Stc}$ B and B $\leq_{Stc}$ C, then A $\leq_{Stc}$ C.

***Proof*:**

Since A $\leq_{Stc}$ B and B $\leq_{Stc}$ C, then by Remark (1.3) (1), A $\leq_c$ B and B $\leq_c$ C. this implies that A $\leq_c$ C, [6, Proposition (1.5), P.18]. And by Proposition (1.7), A is an St-closed submodule in C.

In a similar proof of Proposition (1.9), and by using Remark (1.8) instead of Proposition (1.7) we can prove the following.

Proposition (1.10):

Let M be a fully prime module, and let (0) ≠A $\leq_{Stc}$ B and B $\leq_{Stc}$ M, then A $\leq_{Stc}$ M.

The following remarks verify the hereditary of St-closed property between two submodules of an R-module M.

Remark (1.11):

Let A and B are submodules of an R-module M such that A ≤ B ≤ M. If B is an St-closed submodule in M, then A need not be St-closed submodule in M. For example; the Z-module Z is an St-closed submodule of Z and 2Z ≤ Z, while 2Z is not St-closed submodule in Z.

Remark (1.12):

If A and B are submodules of an R-module M such that A ≤ B ≤ M. If A is an St-closed submodule in M, then B need not be St-closed submodule in M. For example; the Z-module Z and the submodules A = (0) and B = 2Z. Note that (0) is an St-closed submodule in Z, but 2Z is not St-closed submodule in Z, since 2Z is a semi-essential submodule of Z.

Proposition (1.13):

If every submodule of M is an St-closed, then every submodule of M is a direct summand of M.

***Proof*:**

Since every submodule of M is an St-closed, and by Remarks (1.3) (1), every St-closed submodule is a closed, so every submodule of M is a closed. Hence the result follows from [8, Exercises (6- c), P.139].

It is well known that the intersection of two closed submodules need not be closed submodule for example: Consider the Z-module $M = Z \oplus Z_2$, If we take $A = < (1, \bar{0}) >$ and $B = < (1, \bar{1}) >$, it is clear that both of them are direct summands of M, so they are closed in M. But $A \cap B = < (2, \bar{0}) >$ and $(A \cap B) \leq_e B$, that is $A \cap B$ is not closed in M [6, Example (1.6), P.19]. However, we have the following.

Proposition (1.14):

Let A and B be St-closed submodules in an R-module M, then $A \cap B$ is an St-closed submodule in M.

***Proof*:**

Let $L \leq M$ such that $A \cap B \leq_{sem} L \leq M$. By [2, Corollary (6), P.49] $A \leq_{sem} L$ and $B \leq_{sem} L$. Since A and B are St-closed submodules in M, then $A = L = B$, hence $A \cap B = L$.

Proposition (1.15):

Let M be an R-module, and let A and B be submodules of M such that $A \leq B \leq M$. If A is an St-closed submodule in M, then A is an St-closed submodule in B.

***Proof*:**

Suppose that $A \leq_{sem} L \leq B$, so $L \leq M$. But A is an St-closed submodule in M, therefore $A = L$.

Corollary (1.16):

Let A and B be submodules of an R-module M. If $A \cap B$ is an St-closed submodule in M, then $A \cap B$ is an St-closed submodule in A and B.

Corollary (1.17):

If N and K are St-closed submodules in an R-module M, then N and K are St-closed submodules in $N + K$.

***Proof*:**

Since $N \leq N + K \leq M$, so by Proposition (1.15) we are done.

We can proof the following proposition by using [12, Lemma (1.15)]. In fact this Lemma in [12] is true when we instead the condition "fully prime" by the condition "fully essential".

Proposition (1.18):

Let $M = M_1 \oplus M_2$ be a fully essential R-module where M_1 and M_2 be submodules, and let A and B be nonzero submodules of M_1 and M_2 respectively. If A and B are St-closed submodules in M_1 and M_2 respectively. Then $A \oplus B$ is an St-closed submodule in $M_1 \oplus M_2$, provided that ann M_1 + ann $M_2 = R$.

***Proof*:**

Assume that $A \oplus B \leq_{sem} L \leq M$. Since ann M_1 + ann $M_2 = R$, so by the same proof of [1, Proposition (4.2)], $L = L_1 \oplus L_2$, where $L_1 \leq M_1$ and $L_2 \leq M_2$. Therefore $A \oplus B \leq_{sem} L_1 \oplus L_2$, and by [12, Lemma (1.15)], $A \leq_{sem} L_1$ and $B \leq_{sem} L_2$. But both of A and B are St-closed submodules in M. So that $A = L_1$ and $B = L_2$, hence $A \oplus B = L_1 \oplus L_2$.

Proposition (1.19):

Let $M = M_1 \oplus M_2$ be an R-module where M_1 and M_2 be submodules of M, and let A, B be St-closed submodule in M_1 and M_2 respectively. Then $A \oplus B$ is an St-closed submodule in $M_1 \oplus M_2$, provided that aan M_1 + ann $M_2 = R$. And all semi essential extensions of $A \oplus B$ are fully essential modules.

***Proof*:**

Assume that $A \oplus B \leq_{sem} L \leq M$. By the same argument of Proposition (1.18) we have $A \oplus B \leq_{sem} L_1 \oplus L_2$, where $L = L_1 \oplus L_2$. Since L is a fully essential module, then $A \oplus B \leq_e L_1 \oplus L_2$, this implies that $A \leq_e L_1$ and $B \leq_e L_2$. It is clear that both of A and B are closed submodules in M, thus $A = L_1$ and $B = L_2$, hence $A \oplus B = L_1 \oplus L_2$.

Theorem (1.20):

Let $M = M_1 \oplus M_2$ be a fully prime R-module where M_1 and M_2 be submodules of M and let A, B be nonzero submodules of M_1 and M_2 respectively. Then $A \oplus B$ is an St-closed submodule in $M_1 \oplus M_2$ if and only if A and B are St-closed submodules in M_1 and M_2 respectively.

***Proof*:**

⇒) Assume that $A \leq_{sem} K \leq M_1$. Since $B \leq_{sem} B$, we can easily show that $K \oplus B$ is a fully prime module. In fact if X is a proper submodule of $K \oplus B$, and since M is a fully prime module, then X is a prime submodule of M. By [7, Lemma (3.7)], X is a prime submodule of $K \oplus B$, and by [12, Lemma (1.15)], $A \oplus B \leq_{sem} K \oplus B \leq M$. But $A \oplus B \leq_{Stc} M$, thus $A \oplus B = K \oplus B$, that is

A = K. In similar way we can prove that $B \leq_{Stc} M$.

⇐) Since in a fully prime module the St-closed submodule and closed submodule are equivalent, so the result follows from [6, Exercises (15), P.20].

Recall that the prime radical of an R-module M is denoted by rad(M), and it is the intersection of all prime submodules of M [10].

Proposition (1.21):

Let f: M ⟶ M' be an R-epimorphism from an R-module M to an R-module M', and let B be a submodule of M such that ker f ⊆ rad(M) ∩ B. If B is an St-closed submodule in M then f(B) is an St-closed submodule in M'.

***Proof*:**

Let K' be a submodule of M' such that $f(B) \leq_{sem} K' \leq M'$. Since ker f ⊆ rad(M), then $f^{-1}f(B) \leq_{sem} f^{-1}(K') \leq M$ [2]. We can easily show that $f^{-1}f(B) = B$ since ker f ⊆ B. This implies that $B \leq_{sem} f^{-1}(K')$. But B is an St-closed submodule in M, then $B = f^{-1}(K')$. Since f is epimorphism so f(B) = K', and we are done.

Corollary (1.22):

Let A and B be submodules of an R-module M, such that A ⊆ rad(M) ∩ B. if B is an St-closed submodule in M, then $\frac{B}{A}$ is an St-closed submodule in $\frac{M}{A}$.

Recall that a singular submodule defined by $Z(M) = \{x \in M: ann(x) \leq_e R\}$. If Z(M) = M, then M is called the singular module. If Z(M) = 0 then M is called a nonsingular module, [6]. A submodule N of an R-module M is called y-closed submodule of M, if $\frac{M}{N}$ is a nonsingular module [6, P.42]. We cannot find a direct relation between St-closed and y-closed submodules. However, under some conditions we can find some cases of this relationship as the following proposition shows.

Proposition (1.23):

If M is a fully prime R-module, then every nonzero y-closed submodule is an St-closed submodule.

***Proof*:**

Let A be a nonzero y-closed submodule in M, then by [9, Remarks and Examples (2.1.1) (3)], A is a closed submodule in M and by Remark (1.8), A is an St-closed submodule in M.

Proposition (1.24):

Let M be a nonsingular R-module, if a submodule N of M is an St-closed, then N is a y-closed submodule.

***Proof*:**

Let N be an St-closed submodule in M, by Remarks (1.3) (1) N is a closed submodule in M. But M is a nonsingular module, so by [9, Proposition (2.1.2)], N is a y-closed submodule of M.

Another proof:

Assume that M is a nonsingular R-module, and let N be an St-closed submodule in M. Let $Z(\frac{M}{N}) \equiv \frac{B}{N}$, where B is a submodule of M with $N \leq B$. Clearly $\frac{B}{N}$ is a singular module. Now $N \leq B$ and M is a nonsingular module, therefore B is a nonsingular submodule of M. Then by [6, Proposition (1.21), P.32], $N \leq_e B$, hence $N \leq_{sem} B$. But A is an St-closed submodule in M, thus N = B, and $Z(\frac{M}{N}) = (0)$. So $\frac{M}{N}$ is a nonsingular module, and by the definition of y-closed submodule, N is a y-closed submodule in M.

Theorem (1.25):

Let M be a fully prime R-module, and let N be a nonzero submodule of M. Consider the following statement:

1. N is a y-closed submodule.
2. N is a closed submodule.
3. N is an St-closed submodule.

Then (1) ⇒ (2) ⇔ (3), and if M is a nonsingular module, then (3) ⇒ (1)

***Proof*:**

(1) ⇒ (2) [9, Remarks and Examples (2.1.1), 3]

(2) ⇔ (3) Since M is a fully prime module then by, Remark (1.8), N is an St-closed submodule. The converse is clear.

(3) ⇒ (1) Since M is a nonsingular module, then by Proposition (1.24), N is a y-closed submodule.

S_2: St-closed submodules in multiplication modules

In this section we study the behavior of the St-closed submodules in the class of multiplication modules. Also we study the hereditary property of the St-closed submodules between R-modules and R itself.

Recall that An R-module M is called multiplication module, if every submodule N of M is of the form IM for some ideal I of R [4]. Recall that a nonzero prime submodule N of an R-module M is called minimal prime submodule of M if whenever P is a nonzero prime submodule of M such that $P \subseteq N$, then P = N [5].

Proposition (2.1):

Let M be a faithful and multiplication R-module, and let N be a nonzero prime submodule of M. If N is an St-closed submodule in M, then N is a minimal prime submodule of M.

***Proof*:**

Suppose that N is not minimal prime submodule of M. By [2, Prop(3), P.53], N is a semi-essential submodule of M. But N is an St-closed, thus N = M. On the other hand N is a prime submodule that is N must be a proper submodule of M, so we get a contradiction.

Proposition (2.2):

Let M be a nonzero multiplication R-module with only one nonzero maximal submodule N, then N cannot be St-closed submodule in M.

***Proof*:**

Assume that N is an St-closed submodule in M, so by [11, Proposition (2.13)] $N \leq_{sem} M$. By Examples and Remarks (1.2) (4) N = M, but this contradicts with a maximality of N, therefore N is not St-closed submodule in M.

Remark (2.3):

In Proposition (2.2), we get the same result when we replace the condition "nonzero multiplication" by the condition "finitely generated", and by using [11, Proposition (2.14)] instead of [11, Proposition (2.13)].

Proposition (2.4):

Let M be a faithful and multiplication module such that M satisfies the condition (*), if I is an St-closed ideal in J then IM is an St-closed submodule in JM.

Condition (*): For any R-module M and any ideals P and K of R such that P is a prime ideal of K, implies that PM is a prime submodule of KM.

***Proof*:**

Assume that $IM \leq_{sem} L \leq JM$. We have to show that IM = L. Since M is a multiplication module, then L = TM for some ideal T of R. Now $IM \leq_{sem} TM \leq JM$, since M is a faithful and multiplication module and satisfying the condition (*), so by [11, Proposition (2.10)] $I \leq_{sem} T \leq J$. But I is an St-closed ideal in J, then I = T. This implies that IM = TM = L, hence IM is an St-closed submodule in JM.

Proposition (2.5):

Let M be a finitely generated, faithful and multiplication module. If IM is an St-closed submodule in JM, then I is an St-closed ideal in J.

***Proof*:**

Assume that $I \leq_{sem} E \leq J$, then by [11, Proposition (2.11)] $IM \leq_{sem} EM \leq JM$. Since IM is St-closed in JM, then IM = EM. This implies that I = E, [5, Theorem (3.1)]. Thus I is an St-closed submodule in J.

From Proposition (2.4) and Proposition (2.5) we get the following theorem.

Theorem (2.6): Let M be a finitely generated, faithful and multiplication module such that M satisfies the condition (*), then I is an St-closed ideal in J if and only if IM is an St-closed submodule in JM.

Corollary (2.7):

Let M be a finitely generated, faithful and multiplication R-module, and let N be a submodule of M. If M satisfies the condition (*), then the following statements are equivalent:

1. N is an St-closed submodule in M.
2. $(N :_R M)$ is an St-closed ideal in R.
3. N = IM for some St-closed ideal I in R.

***Proof*:**

(1) ⇒ (2) Assume that N is an St-closed submodule in M. Since M is a multiplication module, then $N = (N :_R M)\, M$ [5]. Put $(N :_R M) \equiv I$, so we get IM is an St-closed submodule in M. By Theorem (2.6), I is an St-closed ideal in R.

(2) ⇒ (3) Since M is a multiplication module, then $N = (N :_R M)\, M$ [5], and we are done.

(3) ⇒ (1) Since I is an St-closed ideal in R, so by Theorem (2.6), IM = N is an St-closed submodule in RM = M.

S3:Chain condition on St-closed submodules

In this section we study the chain condition on St-closed submodules, we give some results and examples about this concept. We start by the following definitions.

Definition (3.1):

An R-module M is said to have the ascending chain condition of St-closed submodules (briefly ACC on St-closed submodules), if every ascending chain $A_1 \subseteq A_2 \subseteq \ldots.$ of St-closed submodules in M is finite. That is there exists $k \in Z_+$ such that $A_n = A_k$ for all $n \geq k$.

Definition (3.2):

An R-module M is said to have the descending chain condition of St-closed submodules (briefly DCC on St-closed submodules), if every descending chain $A_1 \supseteq A_2 \supseteq \ldots.$ of St-closed submodules in M is finite. That is there exists $k \in Z_+$ such that $A_n = A_k$, for all $n \geq k$.

Examples and Remarks (3.3):

1) Every Noetherian (respectively Artinian) module satisfies ACC (DCC) on St-closed submodules.

2) Every uniform modules satisfies ACC on St-closed submodules. In fact in a uniform module, the only St-closed submodules are only M and sometime (0).

3) If M satisfies ACC on closed submodules, then M satisfies ACC on St-closed submodules.

***Proof*:**

let $A_1 \subseteq A_2 \subseteq \ldots.$ be an ascending chain of St-closed submodules of M. Since every St-closed submodule is closed submodule, then A_i is a closed submodule $\forall\, i = 1, 2, \ldots$. By assumption M is satisfies ACC on closed submodule, so that $\exists\, k \in Z_+$ such that $A_n = A_k$ $\forall\, n \geq k$. That is M satisfies ACC on St-closed submodules.

Proposition (3.4):

Let M be a finitely generated, faithful and multiplication R-module. Assume that M satisfies the condition (*), then M satisfies ACC on St-closed submodules, if and only if R satisfies ACC on St-closed ideals.

***Proof*:**

⇒): Let $J_1 \subseteq J_2 \subseteq \ldots$ be an ascending chain of St-closed ideals in R. Since J_i is an St-closed ideal in R, then by Theorem (2.6), $J_i M$ is an St-closed submodule in M $\forall\, i = 1, 2, \ldots$. Note that $J_1 M \subseteq J_2 M \subseteq \ldots$ be an ascending chain of St-closed submodules in M. But M satisfies ACC on St-closed submodules, so $\exists\, k \in Z_+$ such that $J_k M = J_n M$ $\forall\, n \geq k$. But M is a finitely generated, faithful and multiplication module, then $J_k = J_n$ $\forall\, n \geq k$ [5, Theorem (3.1)]. Therefore R satisfies ACC on St-closed ideals.

⇐): Let $A_1 \subseteq A_2 \subseteq \ldots$ be an ascending chain of St-closed submodules in M. Since M is a multiplication module, then $A_i = J_i M$ for some ideal J_i of R $\forall\, i = 1, 2, \ldots$. It is clear that $J_1 M \subseteq J_2 M \subseteq \ldots$, since A_i is an St-closed submodule in M $\forall\, i = 1, 2, \ldots$ and M is a finitely generated, faithful and multiplication module and satisfying the condition (*), so by Theorem (2.6), J_i is an St-closed ideal in R $\forall$i 1,2,… . By [5, Theorem (3.1)], $J_1 \subseteq J_2 \subseteq \ldots$, but R satisfies ACC on St-closed ideals, therefore there exists $k \in Z_+$ such that $J_n = J_k$ $\forall\, n \geq k$, so that $J_n M = J_k M$, for each $n \geq k$, thus $A_n = A_k$ $\forall\, n \geq k$. That is M satisfies ACC on St-closed submodules.

Proposition (3.5):

Let M be a chained R-module, and let A be an St-closed submodule of M. If M satisfies ACC on St-closed submodules, then A satisfies ACC on St-closed submodules.

***Proof*:**

Assume that M satisfies ACC on St-closed submodules and $A_1 \subseteq A_2 \subseteq \ldots$, be ascending chain of St-closed submodules of A. Since A is an St-closed submodule of M, and M is a chained module, so by Corollary (1.6), A_i is an St-closed submodule of M. Hence $A_1 \subseteq A_2 \subseteq \ldots$, be ascending chain of St-closed submodules of M. By assumption there exists $k \in Z_+$ such that $A_n = A_k$ $\forall\, n \geq k$, and we are done.

Proposition (3.6):

Let M be an R-module, and let N be a submodule of M such that $N \subseteq rad(M) \cap H$, where H is any St-closed submodule in M. If $\frac{M}{N}$ satisfies ACC on St-closed submodules, then M is satisfies ACC on St-closed submodules.

***Proof*:**

Let $A_1 \subseteq A_2 \subseteq \ldots$ be an ascending chain of St-closed submodules in M. Since A_i is an St-closed submodule in M, and by assumption $N \subseteq rad(M) \cap A_i$, for each i; i = 1, 2, … so by Corollary (1.22), we get $\frac{A_i}{N}$ is an St-closed submodule in $\frac{M}{N}$ for each i; i = 1, 2, … . Hence $\frac{A_1}{N} \subseteq \frac{A_2}{N} \subseteq \ldots$ be ascending chain of St-closed submodules in $\frac{M}{N}$. Since $\frac{M}{N}$ is satisfied ACC on St-closed submodules, so there exists $k \in Z_+$ such that $\frac{A_n}{N} = \frac{A_k}{N}$ $\forall\, n \geq k$. So that $A_n = A_k$ and we get the result.

Proposition (3.7):

Let $M = M_1 \oplus M_2$ be a fully essential R-module, where M_1 and M_2 are submodules. If M satisfies ACC on St-closed submodules, then M_1 (or M_2) satisfies ACC on nonzero St-closed submodules, provided that ann M_1 + ann M_2 = R.

***Proof*:**

Let $A_1 \subseteq A_2 \subseteq \ldots$, be ascending chain of nonzero St-closed submodules of M_1. If M_2 is equal to zero then $M = M_1$, and this implies that M_1 satisfies ACC on nonzero St-closed submodule. Otherwise, since A_i is a nonzero St-closed submodule in M_1, and M_2 is an St-closed submodule in M_2, So by Proposition (1.18), $A_i \oplus M_2$ is an St-closed submodule in M $\forall i = 1,2,\ldots$. Since M satisfies ACC on St-closed submodules, then there exists $k \in Z_+$ such that $A_n \oplus M_2 = A_k \oplus M_2$ $\forall\, n \geq k$. Thus $A_n = A_k, \forall\, n \geq k$. Similarity for M_2.

The converse of Proposition (3.7) is true when every closed submodule of M is fully invariant as the following proposition shows.

Proposition (3.8):

Let $M = M_1 \oplus M_2$ be an R-module, where M_1 and M_2 are St-closed submodules in M. If M_i satisfies ACC on nonzero St-closed submodules, for each i; i = 1, 2. Then M satisfies ACC on nonzero St-closed submodules, provided that every St-closed submodule of M is a fully invariant.

***Proof*:**

Assume that $A_1 \subseteq A_2 \subseteq \ldots$ is an ascending chain of nonzero St-closed submodules in M, and let $\pi_i : M \to M_i$ be the projection maps for each $j \in J$ where J = 1, 2, … . We claim that $A_j = (A_j \cap M_1) \oplus (A_j \cap M_2)$. To verify that, let $x \in A_j$ then $x = m_1 \oplus m_2$, where $m_1 \in M_1$ and $m_2 \in M_2$. Since A_j is an St-closed submodule of M for each $j \in J$, and by our assumption, A_j is a fully invariant which implies that $\pi_1(x) = m_1 \in A_j \cap M_1$ and $\pi_2(x) = m_2 \in A_j \cap M_2$. So $x \in (A_j \cap M_1) \oplus (A_j \cap M_2)$. Thus $A_j \subseteq (A_j \cap M_1) \oplus (A_j \cap M_2)$. But $(A_j \cap M_1) \oplus (A_j \cap M_2) \subseteq A_j$, therefore $A_j = (A_j \cap M_1) \oplus (A_j \cap M_2)$. Note that A_j and M_i are St-closed submodule in M, so by Proposition (1.14), $A_j \cap M_i$ is an St-closed submodule in M. Since $A_j \cap M_i \leq M_i \leq M$, then by Proposition (1.15), $A_j \cap M_i$ is an St-closed submodules in M_i for each i = 1, 2 and j = 1, 2, … . We can easily show that $(A_j \cap M_i) \neq (0)$ for each j = 1,2,… and i = 1,2 . In fact if $A_j \cap M_i = (0)$ for each i = 1, 2 and j = 1, 2, …, then by using $A_j = (A_j \cap M_1) \oplus (A_j \cap M_2)$, we get $A_j = (0)$, which is contradicts with our assumption. That is $A_j \cap M_i$ are nonzero St-closed submodules in M for all i, j. We have the following ascending chain of St-closed submodules in M_i, $(A_1 \cap M_i) \subseteq (A_2 \cap M_i) \subseteq \ldots$, $\forall$ i = 1, 2. But M_i satisfies ACC on nonzero St-closed submodules, then for each i = 1, 2, there exists $k_i \in Z_+$ such that $A_n \cap M_i = A_{k_i} \cap M_i$ $\forall\, n \geq k_i$. Let $k = \max\{k_1, k_2\}$. So $A_n = (A_n \cap M_1) \oplus (A_n \cap M_2) = (A_k \cap M_1) \oplus (A_k \cap M_2) = A_k$ for each $n \geq k$. Thus M satisfies ACC on nonzero St-closed submodules.

Remark (3.9):

We can generalize Proposition (3.8) for finite index I of the direct sum of R-modules.

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الخلاصة

في هذا البحث R هي حلقة أبدالية ذات عنصر محايد وأن M مقاساً أحادياً أيسر على R. ان الهدف الرئيسي من هذا البحث هو دراسة نوع جديد من المقاسات الجزئية (على حد علمنا) أطلقنا عليه أسم المقاسات الجزئية المغلقة من النمط St−, والذي يكون أقوى من مفهوم المقاسات الجزئية المغلقة, أي إن هذا الصنف من المقاسات الجزئية يكون محتوى بشكل فعلي في صنف المقاسات الجزئية المغلقة, حيث انه يقال للمقاس الجزئي N من M بأنه مغلق من النمط St−, إذا كان لا يوجد مقاساً جزئياً فعلياً N في M بحيث إن N يكون شبه جوهري فيه. إن هذا يعني انه إذا وجد مقاساً جزئياً K في M بحيث إن N شبه جوهري في K فإن K = N. يقال للمثالي I في الحلقة R بأنه مقاس جزئي مغلق من النمط St−, إذا كان I مقاساً جزئياً مغلق من النمط St− من المقاس المعرف على الحلقة R. العديد من الخصائص الأساسية درست لهذا النوع من المقاسات الجزئية.