Computer Simulations of Imaging a Dirac Delta Function by a Ground – Based Optical Telescope

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Abstract
Two – dimensional numerical simulations are carried out to study the elements of observing a Dirac point source and a Dirac binary system. The essential features of this simulation are demonstrated in terms of the point spread function and the modulation transfer function. Two mathematical equations have been extracted to present, firstly the relationship between the radius of optical telescope and the distance between central frequency and the cut – off frequency of the optical telescope, and secondly the relationship between the radius of the optical telescope and the average frequency components of the modulation transfer function.
Keywords: Point spread function, modulation transfer function, imaging systems, and optical telescope.

محاكاة حاسوبية لتصوير دالة ديراك بواسطة التلسكوب البصري الارضي

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المستخلص
محاكاة حاسوبية ذات بعدين نفذت لدراسة متطلبات الرصد لمصدر ديراك النقطي ونظام ديراك الثنائي. الصفات الجوهرية للدراسة وضحت في مفاهيم دالة الانتشار النقطية ودالة التضمين الانتقالية. معادلتان رياضيتيان تم استخلاصهما من هذه الدراسة، اولاً العلاقة الرياضية بين نصف قطر التلسكوب البصري والمسافة بين المركز وقطع مركبات التردد للتلسكوب، وثانيا العلاقة الرياضية بين نصف قطر التلسكوب ومعدل مركبات التردد لدالة التضمين الانتقالية.
الكلمات الافتتاحية: دالة الانتشار النقطية، دالة التضمين الانتقالية، انظمة التصوير، والتلسكوب البصري.

1. Introduction:

High frequency components are essential for many optical astronomical applications. These components are necessary features for imaging a very faint object that is close to a sharp bright object or very faint binary system existed at a very long distance from the observer [1]. Finding criteria for imaging a point source is considered one of the most important subjects in astronomy, such as assessing the performance of an optical system [2]. The Rayleigh criterion is considered to be the dominant limit parameter in predicting the performance of an optical imaging system [3]. The performance of a ground optical telescope is severely limited by the point spread function (psf) [4, 5].

In addition to these criteria, ground–based optical imaging systems require perfect correlation between the psf and the radius of the optical telescope aperture. The diffraction of electromagnetic waves causes an optical system to behave as low – pass filter in the formation of an image. The cut – off frequency is directly estimated by the shape and size of the limiting pupil in the optical system. The size of telescope aperture and consequently the value of the cut – off frequency play an important parameter in estimates the psf and equivalently the modulation transfer function (MTF) [6]. The MTF describes the image structure as a function of its spatial frequencies.

The literature involves many works that tackle the problem, that concerning with the limitations of the radius of the optical telescope [7, 8, and 9].

The aim of this paper is to present a quantitative assessment of the quality of psf and MTF of imaging a Dirac delta function and a binary star consisting of two Dirac delta functions using different diameters ground – based optical telescope.

2. Incoherent Imaging Theory:

Consider an extremely distant quasi monochromatic point source (like a Dirac delta function) located on the optical axis of simple imaging system. The Dirac delta function is given by [10].

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{eleswhere} \end{cases} \quad (1)$$

The field distribution incident on the aperture could be expressed as [11].

$$U(\eta,\gamma) = m(\eta,\gamma)e^{i\phi(\eta,\gamma)} \quad (2)$$

where $m(\eta,\gamma)$ and $\phi(\eta,\gamma)$ are the amplitude and phase of the incident wavefront and the variables (η,γ) represent distances in the aperture function. Since the point source is considered to be a Dirac delta function, $m(\eta,\gamma) = 1$ and $\phi(\eta,\gamma) = 0$.

If we assume the aperture of a telescope is a circular function, $p(\eta,\gamma)$, of radius R as given by [12]:

$$p(\eta,\gamma) = \begin{cases} 1 & \text{if } \sqrt{(\eta-\eta c)^2+(\gamma-\gamma c)^2} \le R \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

$(\eta c, \gamma c)$ is the center of two – dimensional array. Then the image of the Dirac delta function which is considered to be the psf of the optical telescope at a certain diameter D. The estimated psf is given by [13]:

$$\text{psf}(x,y) = \left|\text{FT}(p(\eta,\gamma)\right|^2 \quad (4)$$

where FT denotes Fourier transform. The MTF is given by [14]:

$$\text{MTF}(u,v) = \left|\text{FT}(\text{psf}(x,y)\right| \quad (5)$$

The fundamental equation to be used for the formation of an image by an ideal optical system is given by [15].

$$i(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} o(\eta,\gamma)\,psf(x-\eta, y-\gamma)\,d\eta\, d\gamma \quad (6)$$

The above equation is also could be written as:

$$i(x,y) = o(x,y) \otimes psf(x,y) \quad (7)$$

These two equations are equivalent and representing a convolution equation. $i(x,y)$ is the observed or recorded image intensity, $o(x,y)$ is the object intensity, $psf(x,y)$ is the psf or image blurring function caused by an optical system and $\otimes$ denotes convolution operator.

3. Numerical Simulations and Results:

The object to be observed by optical telescope is taken to be a Dirac delta function according to eq. (1). This is done by taking an array of size 512 by 512 pixels of zero values except the central value of this array which is set to one.

$$\delta(x,y) = \begin{cases} 1 & \text{for } x = xc \ \&\ y = yc \\ 0 & \text{eleswhere} \end{cases} \quad (8)$$

where (x_c & y_c) is the index of central point. The absolute Fourier transforms of $\delta(x,y)$ is then estimated and the results are shown in figure (1).

The results demonstrate constant values. This indicates that:

$$f(x,y) \otimes \delta(x,y) = f(x,y) \quad (9)$$

This means that any function convolved with $\delta(x,y)$, the result will be the function itself.

The aperture of optical telescope $p(\eta,\gamma)$ is computed according to eq. (3) by generating an array of size 512 by 512 pixels with a circle of diameter D centered at the middle of this array whose values are ones. The psf and the MTF are computed via eqs. (4) & (5) and the results are shown in figures (2 to 7) respectively.

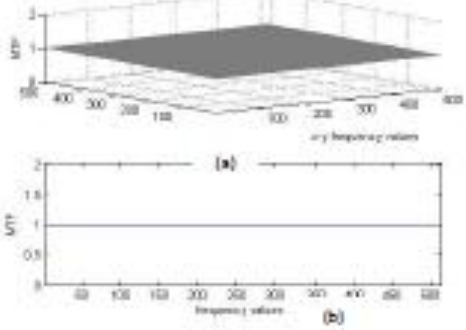

Figure 1: Estimated absolute Fourier transform of a Dirac delta function $\delta(x,y)$
a - Surface plot, and b – Central line.

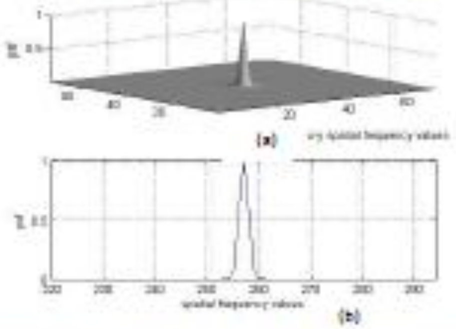

Figure 2: psf at D=240 pixels, a – Surface plot, and b – Central line.

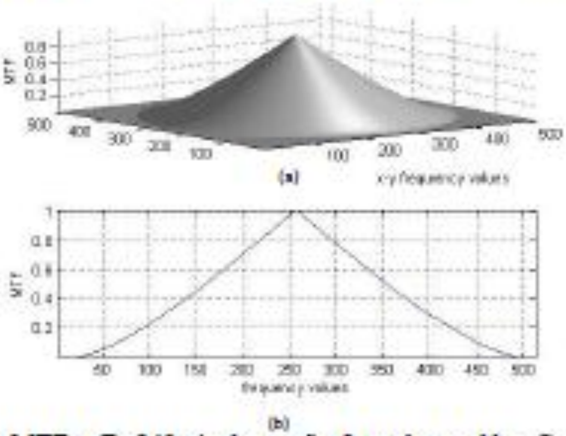

Figure 3: MTF at D=240 pixels, a – Surface plot, and b – Central line.

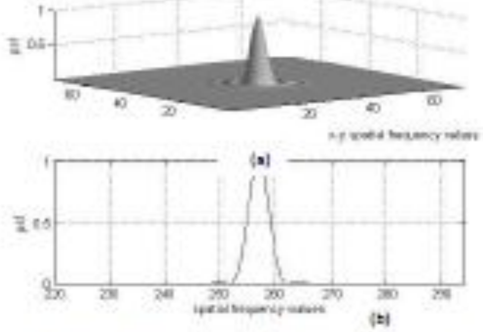

Figure 4: psf at D = 120 pixels, a – Surface plot, and b – Central line.

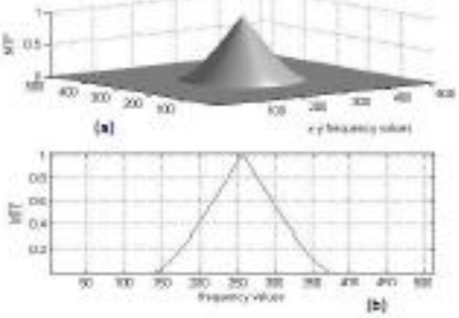

Figure 5: MTF at D = 120 pixels, a – Surface plot, and b – Central line.

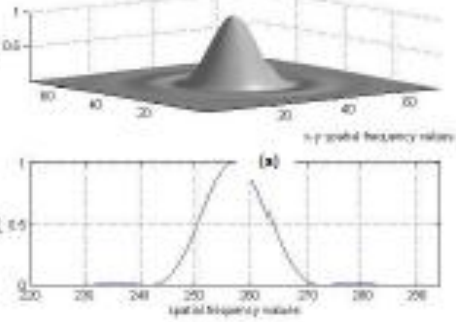

Figure 6: psf at D = 40 pixels, a – Surface plot, and b – Central line.

(b)

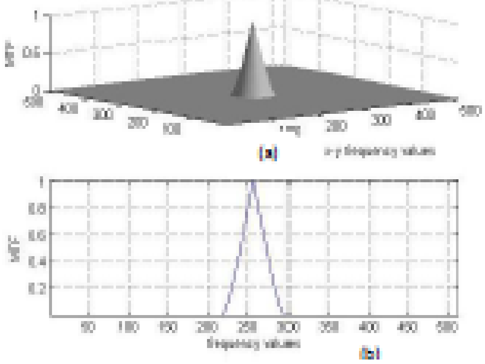

Figure 7: MTF at D= 40 pixels, a – Surface plot, and b – Central line.

The above results demonstrated how the psf and MTF are varies with D. the width of the psf increases and the ripple becomes significance as D increases. The cut – off frequency at different values of D are so clear in MTF plots.

The binary star (bs) is taken to be a two Dirac delta functions and the separation distance is taken to be approximately 11 pixels. This distance is chosen to ensure that the full width at half maximum of the psf for large optical telescope (D = 240 pixels) is far less than 11 pixels as shown in figure (8).

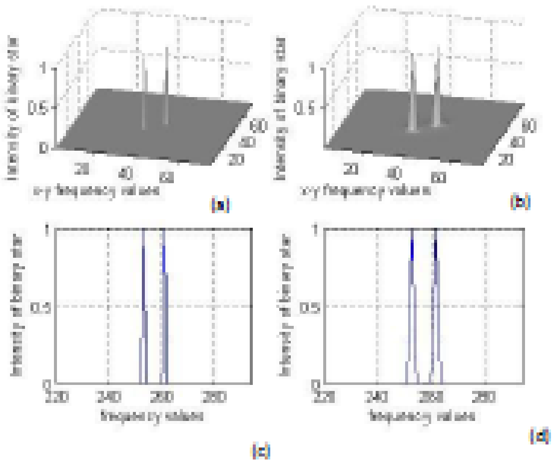

Figure 8: a – Surface plot of original a Dirac binary star, b – Surface plot of a Dirac binary star at D = 240 pixels, c - Central line through (a), and d – Central line through (b).

The power spectrum (pw) of the Dirac binary star is computed by:

$$pw(u,v) = \left| FT \ \ (bs(x,y)) \right|^2 \qquad (10)$$

and the average power spectrum (APS) is given by:

$$APS = \frac{1}{N}\sum_{i=1}^{N} pw_i(u,v) \qquad (11)$$

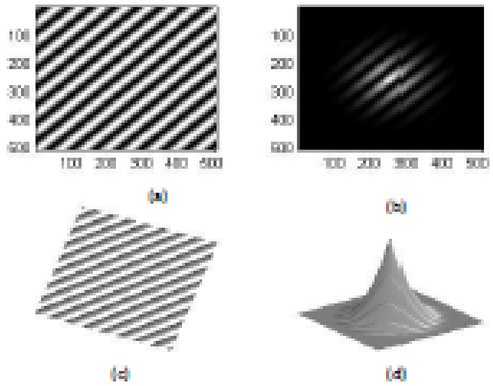

Figure 9: a - Original pw of a Dirac binary star, b – pw of a Dirac binary star at D=240 pixels, c – Surface plot through (a), and d – Surface plot through (b).

The original power spectrum (fig. 9 - a,c) illustrate that the pw reach the boarder of the array and no any attenuation is noticed. This is due to no any blurring that taking place.

The autocorrelation (Aut) of the observed image by the optical telescope has been computed by the following equation:

$$Aut(x,y) = \left|FT^{-1}(pw)\right| \qquad (12)$$

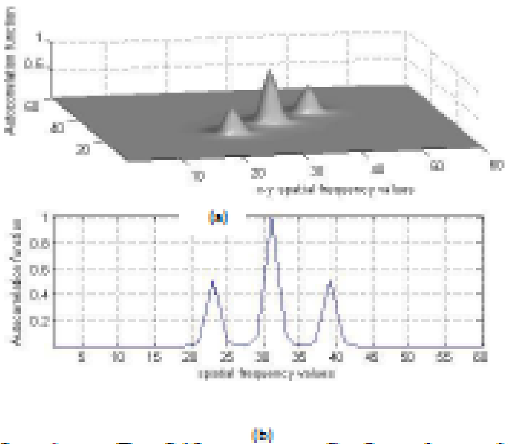

Figure 10: Aut function at D = 240 pixels, a – Surface plot, and b – Central line.

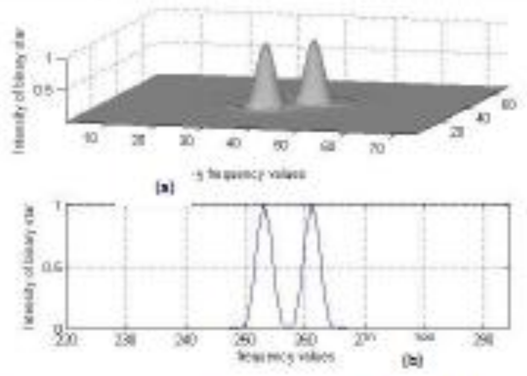

Figure 11: Dirac binary star at D = 120 pixels, a – Surface plot, and b – Central line.

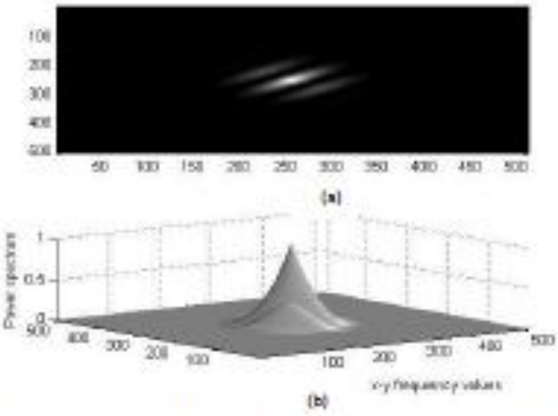

Figure 12: pw of Dirac binary star at D = 120 pixels, a – Image section, and b – Surface plot.

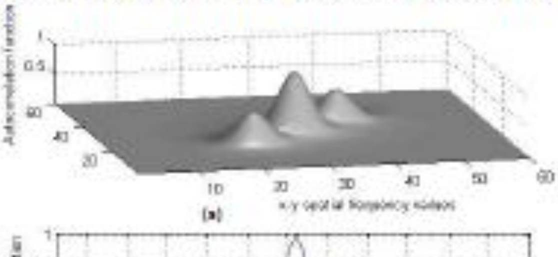

Figure 13: Aut function at D = 120 pixels, a – Surface plot, and b – Central line.

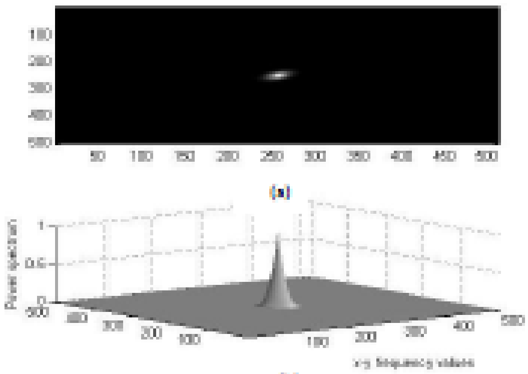

Figure 14: pw of Dirac binary star at D = 4(. s a – Image section, and b – Surface plot.

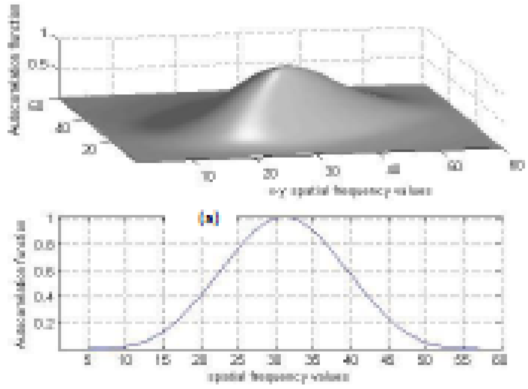

Figure 15: Aut function at D = 40 pixels urface plot, and b – Central line.

The peaks beyond the central spike in pw are attenuated as R increases. The two peaks beyond the central spike in Aut becomes wider as R increases and disappear at D = 40 pixels.

The distance (d) between the central point of MTF and the cut – off frequency is computed with respect to different values of R as shown in the circle points of figure (17 – a). The solid line of this figure represents the least square fitting.
The average frequency components of the MTF (AFM) also are computed according to following equation:

$$AFC = \int_{1}^{M}\int_{1}^{N} MTF(u,v)\,du\,dv \qquad (13)$$

The results illustrated in figure (16 – b). The circle points are the computed data and the solid line is the fitted data.

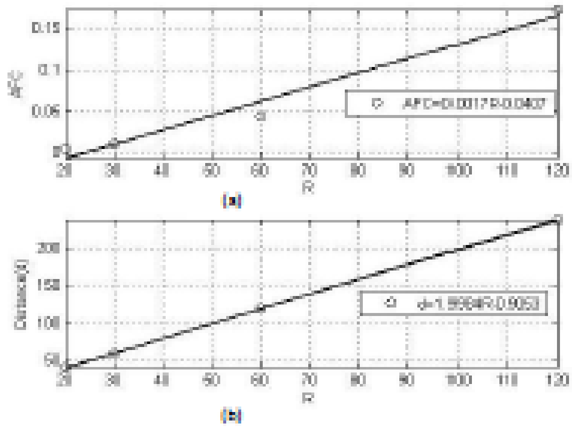

Figure 16: a – AFC as a function of R, b – Distance (d) as a function of R.

APS is computed at different values of R as shown in figure (17).

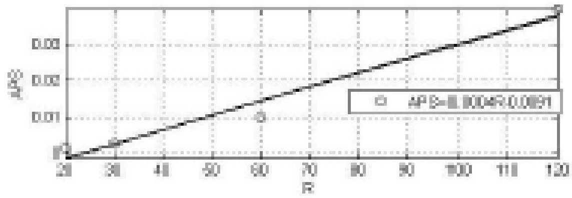

Figure 17: APS as a function of R.

4. Conclusions:

The conclusions that could be drawn from these simulations are as follows:

1. The average frequency components of MTF (AFC) are linearly proportional with the radius of telescope (R) and obey the mathematical equation $AFC = 0.0017R - 0.0407$.
2. The distance between the central frequency (d) and the cut – off frequency is also linearly proportional with R and obey mathematical equation $d = 1.9984R - 0.9053$.
3. The power spectrum of a two dimensional Dirac delta function is fringes of constant brightness and its extent goes to infinity while with optical telescope, this fringes becomes narrow and the brightness becomes smaller as we go away from the center of this fringes.
4. The average power spectrum (APS) is linearly proportional with R and obeys the mathematical equation $APS = 0.0004R - 0.0091$.

5. References:

1. Shu, X; Gao, C. and Ahuja, N. 2012. Aperture Access and Manipulation for Computational Imaging. Computer Vision and Understanding Image Journal, vol.116, pp. 222 – 237.
2. Carlotti, A; Vanderbei, R. and Kasdin, N. J. 2011. Optimal Pupil Apodization for Arbitrary Apertures for High Contrast Imaging, Optics Express, vol.19, No. 27. pp. 26796-26809.
3. Rodrigues, G. M. 2010. Adaptive Optics with Segmented Deformable Bimorph Mirrors, Ph.D. Thesis, Department of Mechanical Engineering and Robotics. University of Libre De Bruxelles. pp.20-21.
4. Friedman, S; Fung, G; Siewersen, J. and Tsui, B. 2013. A Simple Approach to Measure Computed Tomography (CT) Modulation Transfer function (MTF) and Noise – Power Spectrum (NPS) using the American College of Radiology (ACR) Accreditation Phantom. Medical Physics J., vol. 40. No. 5. pp: 51967_1 – 51967_9.

5. Prato, M; Cavicchioli, R; Zanni, L; Boccacci, P. and Bertero, M. **2012**. Efficient Deconvolution Methods for Astronomical Imaging: Algorithms and IDL – GPU Codes. Astronomy & Astrophysics J. vol, 593. pp. A133 – A146.
6. Sheely, C; McCrady, N. and Graham, J. **2006**. Constraining the Adaptive Optics Point – Spread Function in Crowded Fields: Measuring Photometric Aperture Corrections. 2006, Astrophysics J. vol, 647. pp.1517 – 1530.
7. Galicher, R; Baudoz, P. and Baudrand, J. **2011**. Multi-Stage Four-Quadrant Phase Mask: Achromatic Coronagraph for Space-Based and Ground-Based Telescopes. A & A, vol, 530. pp. A43.
8. Mallbet, F; Yu, J. and Shao, M. **1995**. High Dynamic Range Imaging using a Deformable Mirror for Space Corongraphy. PASP, vol, 107. pp. 386-398.
9. Femimor, E. **1980**. Coded Aperture Imaging: The Modulation Transfer Function for Uniformly Redundant Arrays. Applied Optics, vol. 19, No. 14. pp. 2465 – 2471.
10. Martinache, F; Guyon, O; Pluzhnik, E. and Galicher,R. **2006**. Exoplanet Imaging with a Phase-Induced Amplitude Apodization Coronograph II. Performance. The Astrophysical Journal, vol, 639. pp. 1129-1137.
11. Mohammed, A.T. and Jallod, E. Uday. **2013**. Numerical Simulations of Imaging Extrasolar Planets using Circular and Square Apodized Apertures. J. Baghda for Sci., vol, 10. No.4. pp 1232-1241.
12. Sukumar, V; Hess, H; Noren, K; Donohoe, G. and Ay, S. **2008**. Imaging System MTF–Modeling with Modulation Functions, 34th annual Conference of IEEE, 10–13 Nov. Orlando, pp. 1748 – 1753.
13. Mohammed, A.T. **2006**. Quantitative Assessment of Apodized Aperture, Iraqi Journal of Science, vol, 26 No. 1. pp196 – 202.
14. Hassan, Fouad N. **2011**. Imaging Binary Stars in the Presence of Defocusing, Iraqi Journal of Science, vol. 52, No.4, pp.519 – 524.
15. Mohammed, A.T. **2007**. Limits of the Efficiency of Imaging with Obstructing Apertures. SQU:Sultan Qaboos University Journal for Science, vol, 12. No.1.pp 67-74.